

YAMABE FLOW AND ADM MASS ON ASYMPTOTICALLY FLAT MANIFOLDS

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ABSTRACT. In this paper, we investigate the behavior of ADM mass and Einstein-Hilbert functional under the Yamabe flow. Through studying the Yamabe flow by weighted spaces, we show that ADM mass and Einstein-Hilbert functional are well-defined and monotone non-increasing under the Yamabe flow on n -dimensional, $n \geq 3$, asymptotically flat manifolds. In the case of dimension $n = 3$ or 4 , we obtain that the ADM mass is invariant under the Yamabe flow and the Yamabe flow is the gradient flow of Einstein-Hilbert functional on asymptotically flat manifolds

Mathematics Subject Classification (2000): 35J60, 53C21, 58J05

Keywords: Yamabe flow; ADM mass; Einstein-Hilbert functional; Weighted spaces; Asymptotically flat manifolds

1. INTRODUCTION

In 1960, R.Arnowitt, S.Deser, C.Misner ([1],[2],[3]) made a detailed study of isolated gravitational systems, which are modeled by spacetimes that asymptotically approach Minkowski spacetime at infinity, and defined the total mass (the ADM mass) of the gravitational system. They conjectured that the ADM mass is nonnegative if it satisfies the dominant energy condition. The positive mass conjecture was first solved by R.Schoen and S.T.Yau ([32], [33]) using minimal surface and shortly thereafter by E.Witten [38] (see also [29]) using spinors.

The ADM mass of n -dimensional Riemannian manifolds is defined as

$$(1.1) \quad m(g) = \lim_{r \rightarrow \infty} \frac{1}{4\omega_n} \int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj}) * dx_i,$$

where ω_n denotes the volume of $(n-1)$ -sphere with standard metric and S_r denotes the Euclidean sphere with radius r centered at the based point. By the work of R.Bartnik [5], the ADM mass $m(g)$ is independent of choices of the asymptotic coordinates on n -dimensional manifolds when the metric $g \in W_{-\tau}^{2,q}(M)$ (see Definition 3.1) for $q > n$ and $\tau > \frac{n-2}{2}$, and the scalar curvature is nonnegative and integrable.

The higher dimensional Riemannian version of positive mass conjecture states that if (M^n, g) is an n -dimensional asymptotically flat manifold M^n of order $\tau > \frac{n-2}{2}$ and the scalar curvature $R_g \geq 0$, then $m(g) \geq 0$ with equality holds if and only if M^n is isometric to \mathbb{R}^n . R.Schoen [31] gave a proof of his work with S.T.Yau through the use of minimizing hypersurface which proved the Riemannian version of positive mass conjecture for the dimension ≤ 7 . R.Bartnik [5] proved Riemannian version of positive mass conjecture for n -dimensional spin manifolds following E.Witten's methods in [38].

The Yamabe flow is defined by the evolution equation

$$(1.2) \quad \begin{cases} \frac{\partial g}{\partial t} = -Rg & \text{in } M^n \times [0, T), \\ g(\cdot, 0) = g_0 & \text{in } M^n, \end{cases}$$

on an n -dimensional complete Riemannian manifold (M^n, g_0) , $n \geq 3$, where $g(t)$ is a family of Riemannian metrics on the manifold M^n and R is the scalar curvature of the metric

$$g := g(t) = u^{4/(n-2)} g_0,$$

where $n \geq 3$ and $u : M \rightarrow \mathbb{R}^+$ is a positive smooth function on M^n . In the sequence of changing time by a constant scale, (1.2) can be written in the equivalent form

$$(1.3) \quad \begin{cases} \frac{\partial u^N}{\partial t} = L_{g_0} u, & \text{in } M^n \times [0, T), \\ u(\cdot, 0) = 1, & \text{in } M^n, \end{cases}$$

where $N = \frac{n+2}{n-2}$, $L_{g_0} u = \Delta_{g_0} u - a R_{g_0} u$ and $a = \frac{n-2}{4(n-1)}$. The Yamabe flow was proposed by R.Hamilton [19] in the 1980's as a tool for constructing metrics of constant scalar curvature in a given conformal class. Consider the normalized Yamabe flow

$$(1.4) \quad \begin{cases} \frac{\partial g}{\partial t} = (s - R)g & \text{in } M^n \times [0, T), \\ g(\cdot, 0) = g_0 & \text{in } M^n, \end{cases}$$

on closed manifolds, where s is the mean value of R , i.e. $s = \frac{\int_M R d\text{vol}}{\text{vol}(M)}$. R.Hamilton proved that the normalized Yamabe flow (1.4) has a global solution for every initial metric. If the solution $g(t)$ of the Yamabe flow (1.4) converges smoothly as $t \rightarrow \infty$, then the limit metric of $g(t)$ has the constant scalar curvature. Then it would give another approach to Yamabe problem solved by R.Schoen [30]. For this topic, one may see [7], [8], [9], [12], [37], [39] and the references therein for more information.

It is well-known that the ADM mass and positive mass theorem are closed related to the Yamabe problem. In fact, the positive mass theorem plays the key role in R.Schoen's proof of Yamabe problem. Note

that Yamabe flow is the parabolic analogue of the nonlinear elliptic equation in Yamabe problem. So a natural question is that what is the relationship between ADM mass and Yamabe flow? Also notice that the Yamabe flow (1.2) is the gradient flow of Einstein-Hilbert functional $\int_M R d\text{vol}$, which is an important quantity in general relativity, on closed manifolds in a fixed conformal class. Moreover, the boundary term of first variation of Einstein-Hilbert functional on asymptotically flat manifolds is related to the variation of ADM mass (see (3.5)). Therefore, it is natural to ask what is the behavior of Einstein-Hilbert functional under the Yamabe flow (1.2) on asymptotically flat manifolds? We try to answer these questions in this paper.

We remark that geometric flows are one of the powerful tools to study the masses in general relativity and related topics. For example, Huisken and Ilmanen [21] proved the Riemannian-Penrose conjecture in single black hole case by using inverse mean curvature flow, in which Hawking mass is monotone increasing. And H.L.Bray [6] proved the Riemannian-Penrose conjecture by using his conformal flow, in which ADM mass is monotone non-increasing. Recently, an interesting work of X.Dai and L.Ma [15] showed that ADM mass is invariant and well-defined on a 3-dimensional asymptotically flat manifold of order $\tau > \frac{1}{2}$ with nonnegative and integrable scalar curvature. They also obtained the similar results hold for higher dimensional asymptotically manifolds with more assumptions. For other works for studying the ADM mass under the Ricci flow, one may see [24] and [27].

Before presenting the main theorem of this paper, we need the following two definitions.

Definition 1.1. A Riemannian manifold M^n , $n \geq 3$, with C^∞ metric g is called asymptotically flat of order τ if there exists a decomposition $M^n = M_0 \cup M_\infty$ (for simplicity we deal only with the case of one end and the case of multiple ends can be dealt with similarly) with M_0 compact and a diffeomorphism $M_\infty \cong \mathbb{R}^n - B(o, R_0)$ for some $R_0 > 0$ satisfying

$$(1.5) \quad g_{ij} - \delta_{ij} \in C_{-\tau}^{2+\alpha}(M)$$

(defined in Definition 3.1) in the coordinates $\{x^i\}$ induced on M_∞ . And the coordinates $\{x^i\}$ are called asymptotic coordinates.

Definition 1.2. We say that $u(x, t)$ is a fine solution of Yamabe flow, $0 \leq t < t_{max}$, on a complete manifold (M^n, g_0) if $0 < \delta \leq |u(x, t)| \leq C$, $0 \leq t \leq T$, $\sup_{M^n \times [0, T]} |\nabla_{g_0} u(x, t)| \leq C$ and $\sup_{M^n \times [0, T]} |Rm(g)|(x, t) \leq C$, where $T < t_{max}$, such that either $\sup_{M \times [0, t_{max})} |Rm| = \infty$ or $t_{max} = \infty$,

where $Rm(g)$ is the Riemannian curvature of the metric $g := g(t) = u^{4/(n-2)}g_0$.

The short time existence of smooth solution to Yamabe flow on noncompact manifold with bounded scalar curvature was obtained by B.L.Chen, X.P.Zhu [14] and Y.An, L.Ma [4]. Based on their work, we shall show the fine solution to Yamabe flow always exists on asymptotically flat manifolds (see Corollary 2.3).

Our main result in this paper is the following

Theorem 1.3. *Let $u(x, t)$, $0 \leq t < t_{max}$, be the fine solution to the Yamabe flow (1.3) in on an n -dimensional asymptotically flat manifold (M^n, g_0) of order τ . Assume that $R_{g_0} \geq 0$. Set $g(t) = u^{\frac{4}{n-2}}g_0$ for $0 \leq t < t_{max}$. Then*

(i) *the integrability condition $R \in L^1$ is preserved under the Yamabe flow (1.3) if either $R_{g_0} = O(r^{-q})$ for $q > n$ and $\tau > 0$ or $R_{g_0} \in L^1$ and $\tau > \frac{n-2}{2}$. Moreover, $R(g(t)) \geq 0$ and the Einstein-Hilbert functional $\int_M R \text{dvol}$ is monotone non-increasing along the Yamabe flow (1.3) if $R_{g_0} \in L^1$ and $\tau > \frac{n-2}{2}$.*

(ii) *ADM mass $m(g(t))$ is well-defined under the Yamabe flow (1.3) if $\tau > \frac{n-2}{2}$ and $R_{g_0} \in L^1$. More precisely, $g_{ij}(x, t) - \delta_{ij} \in C_{-\tau}^{2+\alpha}(M)$ (defined in Definition 3.1).*

(iii) *ADM mass $m(g(t))$ is monotone non-increasing under the Yamabe flow (1.3) if $\tau > \frac{n-2}{2}$ and $R_{g_0} \in L^1$.*

In the case of dimension $n = 3$ or $n = 4$, we obtain that the ADM mass is invariant under the Yamabe flow (1.2) and the Yamabe flow (1.2) is the gradient flow of Einstein-Hilbert functional on asymptotically flat manifolds of order $\tau > \frac{n-2}{2}$.

Theorem 1.4. *Let $u(x, t)$, $0 \leq t < t_{max}$, be the fine solution to the Yamabe flow (1.3) on an 3 or 4-dimensional asymptotically flat manifold (M^n, g_0) of order $\tau > \frac{n-2}{2}$. Assume that $R_{g_0} \geq 0$. Set $g(t) = u^{\frac{4}{n-2}}g_0$ for $0 \leq t < t_{max}$. Then $m(g(t)) \equiv m(g_0)$ and the Yamabe flow is the gradient flow of Einstein-Hilbert functional.*

In this paper, constants will be denoted by C and its dependence on interesting parameters will be noted as appropriate.

2. PRELIMINARIES

We first recall some basic formulas from [12] (see Lemmas 2.2 and 2.4).

Lemma 2.1. [12] *Let $g(t)$, $n \geq 3$, be the solution to the Yamabe flow (1.2), then the scalar curvature of $g(t)$ evolves by*

$$(2.1) \quad \frac{\partial}{\partial t} R = (n-1)\Delta R + R^2.$$

Moreover, if the initial metric g_0 is locally conformally flat, then the evolution equation of the Ricci curvature is

$$(2.2) \quad \frac{\partial}{\partial t} R_{ij} = (n-1)\Delta R_{ij} + \frac{1}{n-2} B_{ij},$$

where

$$B_{ij} = (n-1)|Ric|^2 g_{ij} + nRR_{ij} - n(n-1)R_{ij}^2 - R^2 g_{ij}.$$

We next recall the short time existence of smooth solution to Yamabe flow (1.3) on noncompact manifolds obtained by B.L.Chen and X.P.Zhu [14] and Y.An, L.Ma [4].

Theorem 2.2. [14][4] *If (M^n, g_0) is an n -dimensional complete manifold with bounded scalar curvature, then Yamabe flow (1.3) has a smooth solution on a maximal time interval $[0, t_{max})$ with $t_{max} > 0$ such that either $t_{max} = +\infty$ or the evolving metric contracts to a point at finite time t_{max} .*

We remark that solution $u(x, t)$ to Yamabe flow (1.3) in Theorem 2.2 satisfies $0 < c_1 < u(x, t) < c_2$ for some constant c_1, c_2 on any compact subinterval of $[0, t_{max})$. In fact, the solution $u(x, t)$ to Yamabe flow (1.3) in Theorem 2.2 is obtained by a sequence of approximation solutions $u_m(x, t)$ which solve the following Dirichlet problem for a sequence of exhausting bounded smooth domains

$$(2.3) \quad \begin{cases} \frac{\partial u_m^N}{\partial t} = L_{g_0} u_m, & x \in \Omega_m, \ t > 0, \\ u_m(x, t) > 0, & x \in \Omega_m, \ t > 0, \\ u_m(x, t) = 1, & x \in \partial\Omega_m, \ t > 0, \\ u_m(\cdot, 0) = 1, & x \in \Omega_m, \end{cases}$$

where $\Omega_1 \subset \Omega_2 \subset \dots$. Since $u_m(x, t) = 1$ is bounded on $\partial\Omega_m$, we may assume $u_m(x, t)$ achieves its maximum $\max_{\Omega_m} u_m$ and minimum $\min_{\Omega_m} u_m$ in the interior of Ω_m . Then by the maximum principle, we have

$$\frac{d}{dt} \max_{\Omega_m} u_m^N(t) \leq a \sup_{M^n} |R_{g_0}|^{\frac{n-2}{4}} u_m(t).$$

Therefore we conclude that

$$\max_{\Omega_m} u_m(t) \leq (1 + \frac{n-2}{(n-1)(n+2)} \sup_{M^n} |R_{g_0}|^{\frac{n-2}{4}}).$$

and

$$\min_{\Omega_m} u_m(t) \geq \left(1 - \frac{n-2}{(n-1)(n+2)} \sup_{M^n} |R_{g_0}| t\right)^{\frac{n-2}{4}}.$$

We see that $u_m(t)$ has an uniformly upper bound on $[0, t_{\max})$ for $t_{\max} < \infty$ and uniformly positive lower bound on $[0, \frac{(n-1)(n+2)}{2(n-2) \sup_{M^n} |R_{g_0}|}]$. We then consider

$$(2.4) \quad \begin{cases} \frac{\partial u_m}{\partial t} = L_{g_0} u_m, & x \in \Omega_m, \quad t > t_0, \\ u_m(x, t) > 0, & x \in \Omega_m, \quad t > t_0, \\ u_m(x, t) = 1, & x \in \partial\Omega_m, \quad t > t_0, \\ u_m(\cdot, t_0) = u_0^m(x), & x \in \Omega_m, \end{cases}$$

where $0 < \delta \leq u_0^m(x) \leq C, x \in \Omega_m$. Again by the maximum principle, we have

$$\min_{\Omega_m} u_m(x, t) \geq \left(\delta^{\frac{4}{n-2}} - \frac{n-2}{(n-1)(n+2)} \sup_{M^n} |R_{g_0}| (t - t_0)\right)^{\frac{n-2}{4}}.$$

Then we get that there exists a constant

$$(2.5) \quad \Delta T(\delta, \sup_{M^n} |R_{g_0}|) = \frac{(n-1)(n+2)\delta^{\frac{4}{n-2}}}{2(n-2) \sup_{M^n} |R_{g_0}|}$$

such that the Dirichlet problem (2.4) has the unique solution satisfying $u_m(x, t) \geq 2^{-\frac{n-2}{4}} \delta$ on $[t_0, t_0 + \Delta T(\delta, \sup_{M^n} |R_{g_0}|)]$. For $t_0 = 0$ and $u_m(\cdot, 0) = 1$ in (2.4), there exists $T_1(1, \sup_{M^n} |R_{g_0}|) = \Delta T_1(1, \sup_{M^n} |R_{g_0}|)$ such that $u_m(x, t) \geq 2^{-\frac{n-2}{4}}$ on $[0, T_1]$. Take $d_1 = \inf_{m, \Omega_m} u_m(x, T_1)$. For $t_0 = T_k$ and $u_0^m(x) = u_m(\cdot, T_{k-1})$ in (2.4), there exists $T_k(d_{k-1}, \sup_{M^n} |R_{g_0}|) = T_{k-1} + \Delta T_{k-1}(d_{k-1}, \sup_{M^n} |R_{g_0}|)$ such that $u_m(x, t) \geq 2^{-\frac{n-2}{4}} d_{k-1}$ on $[T_{k-1}, T_k]$. Take $d_k = \inf_{m, \Omega_m} u_m(x, T_k)$. Then we have either $T_k \rightarrow T < \infty$ or $T_k \rightarrow \infty$. If $T_k \rightarrow T < \infty$, then $\Delta T_k \rightarrow 0$. By (2.5), we see that $d_k \rightarrow 0$ and $T = t_{\max}$ is the blow-up time. This implies the solution $u(x, t)$ to Yamabe flow (1.3) in Theorem 2.2 satisfies $0 < c_1 < u(x, t) < c_2$ for some constants c_1, c_2 on any compact subinterval of $[0, t_{\max})$.

Based on Theorem 2.2 and the remark above, we can show that there exists a fine solution to Yamabe flow (1.3) on asymptotically flat manifolds.

Corollary 2.3. *If (M^n, g_0) is an n -dimensional asymptotically flat manifold of order $\tau > 0$, then Yamabe flow (1.3) has a fine solution $u(x, t)$ on a maximal time interval $[0, t_{\max})$ with $t_{\max} > 0$.*

Proof. By Theorem 2.2, there exists a smooth solution $u(x, t)$ to Yamabe flow (1.3) on a maximal time interval $[0, t_{max})$ with $t_{max} > 0$ such that $0 < \delta \leq |u(x, t)| \leq C$ on any compact time interval of $[0, t_{max})$. Let r_0 be a fixed positive constant. Applying the Krylov-Safonov estimate and Schauder estimate of parabolic equations to (1.3) on $B_{g_0}(p, r_0)$ (see [23]), we have $\sup_{B_{g_0}(p, r_0) \times [0, T]} |\nabla_{g_0}^2 u(x, t)| \leq C$ for $T < t_{max}$, and therefore $\sup_{B_{g_0}(p, r_0) \times [0, T]} |Rm(x, t)| \leq C$, where C is independent of the point p . \square

We also need a slight generalized version of the maximum principle obtained by K.Ecker and G.Huisken (see Theorem 4.3 in [16]), where they consider the maximum principle for the parabolic equation $\frac{\partial}{\partial t} v - \Delta v \leq b \cdot \nabla v + cv$ on noncompact manifolds, where Δ and ∇ depend on $g(t)$. With little more observation, K.Ecker and G.Huisken's maximum principle can be easily generalized to fitting the equation $\frac{\partial}{\partial t} v - \text{div}(a \nabla v) \leq b \cdot \nabla v + cv$. We shall give a proof of Theorem 2.4 in the appendix for sake of convenience for the readers.

Theorem 2.4. *Suppose that the complete noncompact manifold M^n with Riemannian metric $g(t)$ satisfies the uniformly volume growth condition*

$$\text{vol}_{g(t)}(B_{g(t)}(p, r)) \leq \exp(k(1 + r^2))$$

for some point $p \in M$ and a uniform constant $k > 0$ for all $t \in [0, T]$. Let v be a differentiable function on $M \times (0, T]$ and continuous on $M \times [0, T]$. Assume that v and $g(t)$ satisfy

(i) the differential inequality

$$\frac{\partial}{\partial t} v - \text{div}(a \nabla v) \leq b \cdot \nabla v + cv,$$

where the vector field b and the function a and c are uniformly bounded

$$0 < \alpha'_1 \leq a \leq \alpha_1, \quad \sup_{M \times [0, T]} |b| \leq \alpha_2, \quad \sup_{M \times [0, T]} |c| \leq \alpha_3,$$

for some constants $\alpha'_1, \alpha_1, \alpha_2 < \infty$. Here Δ and ∇ depend on $g(t)$.

(ii) the initial data

$$v(p, 0) \leq 0,$$

for all $p \in M$.

(iii) the growth condition

$$\int_0^T \left(\int_M \exp[-\alpha_4 d_{g(t)}(p, y)^2] |\nabla v|^2(y) d\mu_t \right) dt < \infty.$$

for some constant $\alpha_4 > 0$.

(iv) bounded variation condition in metrics

$$\sup_{M \times [0, T]} \left| \frac{\partial}{\partial t} g(t) \right| \leq \alpha_5$$

for some constant $\alpha_5 < \infty$. Then, we have

$$v \leq 0$$

on $M \times [0, T]$.

The following gradient estimate for scalar curvature is needed in the proofs of Theorem 1.4 and Theorem 5.2.

Theorem 2.5. *Let $g(t)$ be the solution to Yamabe flow (1.2) on an n -dimensional manifold (M^n, g_0) . Let $K < \infty$ be positive constants. For each $r > 0$, there is a constant $C(n)$ such that if*

$$(2.6) \quad |Rc(x, t)| \leq K \text{ for } (x, t) \in B_{g_0}(p, r) \times [0, \tau],$$

then

$$|\nabla R(x, t)| \leq C(n)K \left(\frac{1}{r^2} + \frac{1}{\tau} + K \right)^{\frac{1}{2}}$$

for all $x \in B_{g_0}(p, \frac{r}{2})$ and $t \in (0, \tau]$.

Proof. We just proceed as W.X.Shi's gradient estimate ([34], [35]) of Ricci flow. From (2.1) we obtain

$$\frac{\partial}{\partial t} R^2 = (n-1)(\Delta R^2 - 2|\nabla R|^2) + 2R^3,$$

and

$$\begin{aligned} \frac{\partial}{\partial t} |\nabla R|^2 &= (n-1)(\Delta |\nabla R|^2 - 2|\nabla^2 R|^2) + 5R|\nabla R|^2 - 2(n-1)Rc(\nabla R, \nabla R) \\ &\leq (n-1)(\Delta |\nabla R|^2 - 2|\nabla^2 R|^2) + C(n)K|\nabla R|^2, \end{aligned}$$

where $C(n)$ depends only on n . Then we compute

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - (n-1)\Delta \right) ((16K^2 + R^2)|\nabla R|^2) \\ & \leq |\nabla R|^2 (2R^3 - 2(n-1)|\nabla R|^2) \\ & \quad + (16K^2 + R^2)(-2(n-1)|\nabla^2 R|^2 + C(n)K|\nabla R|^2) \\ & \quad + 8(n-1)|R||\nabla R|^2|\nabla^2 R| \\ & \leq -2(n-1)|\nabla R|^4 + 2K^3|\nabla R|^2 - 32(n-1)K^2|\nabla^2 R|^2 \\ & \quad + 17C(n)K^3|\nabla R|^2 + 8(n-1)K|\nabla R|^2|\nabla^2 R|. \end{aligned}$$

Since

$$-\frac{1}{2}|\nabla R|^4 + 8K|\nabla R|^2|\nabla^2 R| - 32K^2|\nabla^2 R|^2 \leq 0,$$

and

$$-\frac{1}{2}|\nabla R|^4 + (2 + 17C(n)K^3)|\nabla R|^2 \leq C'(n)K^6,$$

we have

$$\left(\frac{\partial}{\partial t} - (n-1)\Delta\right)((16K^2 + R^2)|\nabla R|^4) \leq -|\nabla R|^4 + C'(n)K^6.$$

Taking $G = \min\left\{\frac{1}{289}, \frac{1}{C'(n)}\right\} \frac{(16K^2 + R^2)|\nabla R|^2}{K^4}$, we get

$$\left(\frac{\partial}{\partial t} - (n-1)\Delta\right)G \leq -G^2 + K^2.$$

From $\frac{\partial g}{\partial t} = -Rg$, we have

$$e^{-nKT}g_0 \leq g(t) \leq e^{nKT}g_0$$

for $t \in [0, T)$. Following Hamilton's proof of W.X.Shi's gradient estimate for Ricci flow, we can choose the barrier function $H = \frac{BA^2}{\phi^2} + \frac{1}{t} + K$ which defined on $V \times (0, \infty)$, where B, A are the constants only depending on n and $V = \{x \in M \mid \phi(x) > 0\}$ with compact closure in $B_{g_0}(p, r)$. Moreover, $\Lambda = \{(x, t) \in V \times (0, \tau) \mid G(x, t) \geq H(x, t)\}$ is compact and H satisfies $\frac{\partial H}{\partial t} > \Delta H - H^2 + K^2$ as long as $G \leq H$ (see Lemma 6.19 and Lemma 6.20 in [13]). Then Theorem 2.5 follows from $G < H$ on $V \times (0, \tau]$ by maximal principle. \square

As an immediate application to Lemma 2.1, Theorem 2.4 and Theorem 2.5, the following corollary holds.

Corollary 2.6. *Let $g(t)$ be the solution to Yamabe flow (1.2) on an n -dimensional asymptotically flat manifold (M^n, g_0) of order $\tau > 0$. If $R_{g_0} \geq 0$, then $R(g(t)) \geq 0$ for $0 \leq t < t_{max}$.*

Finally, we need the following lemma.

Lemma 2.7. *Let $g(t)$, $0 \leq t \leq T$, be the solution to the Yamabe flow (1.2) on an n -dimensional asymptotically flat manifold (M^n, g_0) of order $\tau > 0$. Suppose $g(t)$ has uniformly bounded curvature. For sufficient large R , there is a smooth positive function f on M such that*

$$f(x) = C_0 >> 1, \text{ for } x \in B_R,$$

and

$$cd_{g(t)} \leq f(x) \leq Cd_{g(t)}.$$

Moreover

$$f \geq C_0, |\nabla_{g(t)} f| \leq C_1, |\Delta_{g(t)} f| \leq C_2,$$

where C_1 and C_2 are the positive constants only depending on the bound of the curvature.

Proof. Choose the smooth increasing function $\zeta(s)$ such that

$$\zeta(s) = \begin{cases} R-1, & s \leq R-1; \\ s, & s \geq R. \end{cases}$$

and

$$|\zeta'| \leq 1, \quad |\zeta''| \leq 2.$$

Let $f(x) = \zeta(r(x))$ and $C_0 = R-1$. Then $f \geq C_0$. Since the curvature is uniformly bounded and g_0 is asymptotically flat, we have

$$|\nabla_{g(t)} f| = |\zeta'| |\nabla_{g(t)} r(x)| \leq C_1,$$

and

$$|\Delta_{g(t)} f| \leq |\zeta''| |\nabla_{g(t)} r(x)|^2 + |\zeta'| |\Delta_{g(t)} r(x)| \leq C_2.$$

□

3. THE BEHAVIOR OF ADM MASS AND EINSTEIN-HILBERT FUNCTIONAL UNDER THE YAMABE FLOW

In this section, we study the behavior of ADM mass and Einstein-Hilbert functional under the Yamabe flow. We first recall the definitions of weighted spaces (see [5] and [18]) for elliptic operators on asymptotically flat manifolds.

Definition 3.1. Suppose (M^n, g) is an n -dimensional asymptotically flat manifold with asymptotic coordinates $\{x^i\}$. Denote $D_x^j v = \sup_{|\alpha|=j} \left| \frac{\partial^{|\alpha|}}{\partial x_{i_1} \cdots \partial x_{i_j}} v \right|$.

Let $r(x) = |x|$ on M_∞ and extend r to a smooth positive function on all of M^n . For $q \geq 1$ and $\beta \in \mathbb{R}$, the weighted Lebesgue space $L_\beta^q(M)$ is defined as the set of locally integrable functions v for which the norm

$$\|v\|_{L_\beta^q(M)} = \begin{cases} \left(\int_M |v|^q r^{-\beta q - n} dx \right)^{\frac{1}{q}}, & q < \infty; \\ \operatorname{ess\,sup}_M (r^{-\beta} |v|), & q = \infty, \end{cases}$$

is finite. Then the weighted Sobolev space $W_\beta^{k,q}(M)$ is defined as the set of functions v for which $|D_x^j v| \in L_{\beta-j}^q(M)$ with the norm

$$\|v\|_{W_\beta^{k,q}(M)} = \sum_{j=0}^k \|D_x^j v\|_{L_{\beta-j}^q(M)}.$$

For a nonnegative integer k , the weighted C^k space $C_\beta^k(M)$ is defined as the set of C^k functions v for which the norm

$$\|v\|_{C_\beta^k(M)} = \sum_{j=0}^k \sup_M r^{-\beta+j} |D_x^j v|$$

is finite. Then the weighted Hölder space $C_\beta^{k+\alpha}(M)$ is defined as the set of functions $v \in C_\beta^k(M)$ for which the norm

$$\|v\|_{C_\beta^{k+\alpha}(M)} = \|v\|_{C_\beta^k(M)} + \sup_{x \neq y \in M} \min(r(x), r(y))^{-\beta+k+\alpha} \frac{|D_x^k v(x) - D_x^k v(y)|}{|x - y|^\alpha}.$$

is finite.

The theory of elliptic operators on the weighted spaces defined in 3.1 was first introduced by Nirenberg and Walker [28], and has been studied by many mathematicians such as Lockhart [25], McOwen [26], Cantor [11], Bartnik [5] and others. In [5], R. Bartnik proved following theorem by using weighted spaces defined in Definition 3.1 (see also Theorem 9.6 in [18]).

Theorem 3.2. (*R. Bartnik*) *If (M^n, g) is an n -dimensional asymptotic flat manifold with $g_{ij} - \delta_{ij} \in C_{-\tau}^{1+\alpha}(M)$ and $R_g \in L^1(M)$, where $\tau > \frac{n-2}{2}$, then ADM mass $m(g)$ defined in (1.1) is independent of the choices of coordinates.*

In order to prove Theorem 1.3, we need show that the Yamabe flow preserves the asymptotically flat condition (1.5) on the asymptotically flat manifold.

Theorem 3.3. *Let $u(x, t)$, $0 \leq t < t_{max}$, be the fine solution to the Yamabe flow (1.3) on an n -dimensional asymptotically flat manifold (M^n, g_0) of order $\tau > \frac{n-2}{2}$. Set $g(t) = u^{\frac{4}{n-2}} g_0$ and $v = 1 - u$. Then $v(x, t) = O(r^{-(\tau+2)})$ and $R(x, t) = O(r^{-(\tau+2)})$. Moreover, $v(x, t) \in C_{-\tau}^{2+\alpha}(M)$ and $g_{ij}(x, t) - \delta_{ij} \in C_{-\tau}^{2+\alpha}(M)$.*

The main tools for the proof of Theorem 3.3 are the weighted spaces for parabolic operators analogous to Definition 3.1 (see Definition 4.1). We will present the proof of Theorem 3.3 in Section 5.

Theorem 3.4. *Let $u(x, t)$, $0 \leq t < t_{max}$, be the fine solution to the Yamabe flow (1.3) on an n -dimensional asymptotically flat manifold (M^n, g_0) of order $\tau > \frac{n-2}{2}$ with asymptotic coordinates $\{x_i\}$. Assume $m(g(t))$ and $m(g_0)$ are the ADM mass defined in (1.1) with same asymptotic coordinates $\{x_i\}$. Then ADM mass $m(g(t))$ is monotone non-increasing on $0 \leq t < t_{max}$.*

Proof. Recall the solution $u(x, t)$ to Yamabe flow (1.3) in Theorem 2.2 is obtained by a sequence of approximation solutions $u_m(x, t)$ which solve the Dirichlet problem (2.4) for a sequence of exhausting bounded smooth domains where $\Omega_1 \subset \Omega_2 \subset \dots$. Without loss of generality, we may assume $\Omega_m = B(o, m)$. Fix time $t_0 > 0$ and set $g_m(t) = u_m^{\frac{4}{n-2}} g_0$. We consider the Dirichlet problem that

$$(3.1) \quad \begin{cases} \frac{\partial g_m}{\partial t} = -Rg_m, & x \in \Omega_m, \ t > t_0, \\ g_m(t) = g_m(t_0), & x \in \partial\Omega_m, \ t > t_0, \end{cases}$$

where $g_m(t) = \tilde{u}_m^{\frac{4}{n-2}}(x, t)g_m(t_0)$. By the uniqueness of the Dirichlet problem, we have $\tilde{u}_m(x, t) = \frac{u_m(x, t)}{u_m(x, t_0)}$. Note that \tilde{u}_m satisfies the following equation

$$(3.2) \quad \begin{cases} \frac{\partial \tilde{u}_m}{\partial t} = L_{g_m(t_0)} \tilde{u}_m, & x \in \Omega_m, \ t > t_0, \\ \tilde{u}_m(x, t) > 0, & x \in \Omega_m, \ t > t_0, \\ \tilde{u}_m(x, t) = 1, & x \in \partial\Omega_m, \ t > t_0, \\ \tilde{u}_m(\cdot, t_0) = 1, & x \in \Omega_m, \end{cases}$$

By the maximal principle that $\tilde{u}_m(x, t) \leq 1$ on Ω_m for $t \geq t_0$. Since $\tilde{u}_m = 1$ on $\partial\Omega_m$, we deduce that $\frac{\partial \tilde{u}_m}{\partial \nu} \geq 0$ on $\partial\Omega_m$, where ν is the outer unit normal vector with respect to $g_m(t_0)$. Now we denote $(\xi_m(t))_i =$

$(g_m(t))_{ij,j} - (g_m(t))_{jj,i}$ and $<, > = g_m(t_0)(,)$. We calculate

$$\begin{aligned}
& \frac{1}{4\omega_n} \int_{S_m} < \xi_m(t), \nu > dS_m(t) \\
&= \frac{1}{4\omega_n} \int_{S_m} \tilde{u}_m(x, t)^{\frac{2n+2}{n-2}} < \xi_m(t_0), \nu > dS_m(t_0) \\
& \quad + \frac{1}{(n-2)\omega_n} \int_{S_m} \tilde{u}_m(x, t)^{\frac{n+4}{n-2}} (\tilde{u}_m(x, t)_{,j} (g_m)_{ij}(t_0) - \tilde{u}_m(x, t)_{,i} g(t_0)_{jj}) \nu^i dS_m(t_0) \\
&= \frac{1}{4\omega_n} \int_{S_m} \tilde{u}_m(x, t)^{\frac{2n+2}{n-2}} < \xi_m(t_0), \nu > dS_m(t_0) \\
& \quad + \frac{1}{(n-2)\omega_n} \int_{S_m} \tilde{u}_m(x, t)^{\frac{n+4}{n-2}} ((n-1)\tilde{u}_m(x, t)_{,i} + \tilde{u}_m(x, t)_{,j} (g_m)_{ij}(t_0) \\
& \quad - \tilde{u}_m(x, t)_{,i} g(t_0)_{jj}) \nu^i dS_m(t_0) - \frac{1}{(n-2)\omega_n} \int_{S_m} (n-1)\tilde{u}_m(x, t)^{\frac{n+4}{n-2}} \frac{\partial \tilde{u}_m(x, t)}{\partial \nu} dS_m(t_0) \\
&\leq \frac{1}{4\omega_n} \int_{S_m} \tilde{u}_m(x, t)^{\frac{2n+2}{n-2}} < \xi_m(t_0), \nu > dS_m(t_0) \\
& \quad + \frac{1}{(n-2)\omega_n} \int_{S_m} \tilde{u}_m(x, t)^{\frac{n+4}{n-2}} ((n-1)\tilde{u}_m(x, t)_{,i} + \tilde{u}_m(x, t)_{,j} (g_m)_{ij}(t_0) \\
& \quad - \tilde{u}_m(x, t)_{,i} g(t_0)_{jj}) \nu^i dS_m(t_0) \\
(3.3) \quad & - \tilde{u}_m(x, t)_{,i} g(t_0)_{jj}) \nu^i dS_m(t_0)
\end{aligned}$$

Note that $0 < \delta \leq |u(x, t)| \leq C$ and $\sup_{M^n \times [0, T]} |\nabla_{g_0} u(x, t)| \leq C$ on time interval $[0, t]$. By Theorem 3.3, we know that $u(x, t) \rightarrow 1$ as $r \rightarrow \infty$ and $u(x, t)_{,i} = O(r^{-(\tau+1)})$ for $\tau > \frac{n-2}{2}$. Since $\tilde{u}_m(x, t) = \frac{u_m(x, t)}{u_m(x, t_0)}$, we have $\tilde{u}(x, t) = \frac{u(x, t)}{u(x, t_0)}$ by letting $m \rightarrow \infty$. Then we have that $\tilde{u}(x, t) \rightarrow 1$ as $r \rightarrow \infty$ and $\tilde{u}(x, t)_{,i} = O(r^{-(\tau+1)})$ for $\tau > \frac{n-2}{2}$. Then taking $m \rightarrow \infty$ in (3.3), we conclude that

$$\begin{aligned}
m(g(t)) &\leq m(g(t_0)) + \frac{1}{(n-2)\omega_n} \lim_{m \rightarrow \infty} \int_{S_m} \tilde{u}(x, t)^{\frac{n+4}{n-2}} ((n-1)\tilde{u}(x, t)_{,i} \\
& \quad + \tilde{u}(x, t)_{,j} (g(t_0))_{ij} - \tilde{u}(x, t)_{,i} g(t_0)_{jj}) \nu^i dS_m(t_0) \\
&= m(g(t_0)) + \frac{1}{(n-2)\omega_n} \lim_{m \rightarrow \infty} \int_{S_m} \tilde{u}(x, t)^{\frac{n+4}{n-2}} ((n-1)\tilde{u}(x, t)_{,i} \\
& \quad + \tilde{u}(x, t)_{,j} (\delta_{ij} + O(r^{-\tau})) - \tilde{u}(x, t)_{,i} (n + O(r^{-\tau})) \nu^i dS_m(t_0) \\
&= m(g(t_0)) + \frac{1}{(n-2)\omega_n} \lim_{m \rightarrow \infty} \int_{S_m} \tilde{u}(x, t)^{\frac{n+4}{n-2}} \tilde{u}(x, t)_{,i} \nu^i O(r^{-\tau}) dS_m(t_0) \\
&= m(g(t_0))
\end{aligned}$$

□

For the mass to be well defined, we need the integrability condition $R \in L^1$ is preserved under the Yamabe flow by Theorem 3.2. It is obtained by proving the Einstein-Hilbert functional is monotone non-increasing under the fine solution to the Yamabe flow (1.3).

Theorem 3.5. *Let $u(x, t)$, $0 \leq t < t_{max}$, be the fine solution to the Yamabe flow (1.3) on an n -dimensional asymptotically flat manifold (M^n, g_0) of order $\tau > \frac{n-2}{2}$. Set $g(t) = u^{\frac{4}{n-2}} g_0$. Assume $\int_M R_{g_0} dvol_{g_0}$ is finite. Then Einstein-Hilbert functional $\int_M R dvol_{g(t)}$ is monotone non-increasing along the Yamabe flow (1.3) on $[0, t_{max})$. More precisely, we have*

$$(3.4) \quad \frac{d}{dt} \int_M R dvol_{g(t)} \leq \left(1 - \frac{n}{2}\right) \int_M R^2 dvol_{g(t)} \leq 0,$$

for $t \in [0, t_{max})$.

Proof. Let $\frac{\partial g_{ij}}{\partial t} = v_{ij}$. Then it follows from the variation of Einstein-Hilbert functional (see (8.9) and (8.11) in [18]) that

$$(3.5) \quad \begin{aligned} \frac{d}{dt} \int_M R dvol_{g(t)} &= \lim_{r \rightarrow \infty} \int_{S_r} \langle \xi, \nu \rangle dS - \int_M v^{ij} G_{ij} dvol_{g(t)} \\ &= \lim_{r \rightarrow \infty} \int_{S_r} (v_{ij,j} - v_{jj,i}) * dx_i - \int_M v^{ij} G_{ij} dvol_{g(t)} \end{aligned}$$

where $\xi_i = (v_{ij,j} - v_{jj,i})(1 + O(r^{-1}))$ and $G_{ij} = Rc_{ij} - \frac{1}{2}Rg_{ij}$ is the Einstein tensor. For sake of convenience for the readers, we give a proof of (3.5) below. By the variation of scalar curvature (see Lemma 2.7 in [13]),

$$\frac{\partial}{\partial t} R = -\Delta(g^{ij}v_{ij}) + \operatorname{div}(\operatorname{div}(v)) - v^{ij}Rc_{ij}.$$

Hence

$$\frac{\partial}{\partial t} (R dvol_{g(t)}) = (-\Delta(g^{ij}v_{ij}) + \operatorname{div}(\operatorname{div}(v)) - v^{ij}(Rc_{ij} - \frac{1}{2}Rg_{ij})) dvol_{g(t)}.$$

Then

$$\begin{aligned}
& \frac{d}{dt} \int_{B(o,r)} R dvol_{g(t)} \\
&= \int_{B(o,r)} (-\Delta(g^{ij}v_{ij}) + \operatorname{div}(\operatorname{div}(v))) dvol_{g(t)} - \int_{B(o,r)} v^{ij} G_{ij} dvol_{g(t)} \\
&= \int_{S_r} \langle \nabla(g^{ij}v_{ij}) - \operatorname{div}(v), \nu \rangle dS - \int_{B(o,r)} v^{ij} G_{ij} dvol_{g(t)} \\
&= \int_{S_r} \langle \xi, \nu \rangle dS - \int_{B(o,r)} v^{ij} G_{ij} dvol_{g(t)}.
\end{aligned}$$

Letting $r \rightarrow \infty$, we get (3.5). Note that $v_{ij} = -Rg_{ij}$ for the Yamabe flow. Since $R(g(t)) = O(r^{-(2+\tau)})$ by Theorem 3.3 and $\tau > \frac{n-2}{2}$, we conclude that R^2 is integrable. Hence

$$\begin{aligned}
& \frac{d}{dt} \int_M R dvol_{g(t)} \\
&= 4\omega_n \frac{d}{dt} m(g(t)) + \lim_{r \rightarrow \infty} \int_{S_r} (g_{ij,j} - g_{jj,i}) \frac{\partial}{\partial t} (*dx_i) + (1 - \frac{n}{2}) \int_M R^2 dvol_{g(t)} \\
&\leq (1 - \frac{n}{2}) \int_M R^2 dvol_{g(t)},
\end{aligned}$$

by Theorem 3.4, $g_{ij,j} - g_{jj,i} = O(r^{-(\tau+1)})$ for $\tau > \frac{n-2}{2}$ by Theorem 3.3, and $\frac{\partial}{\partial t} (*dx_i) = O(r^{-(\tau+2)})$ for $\tau > \frac{n-2}{2}$. \square

Furthermore, if we assume

$$(3.6) \quad R_{g_0} = O(r^{-\sigma}), \quad \sigma \geq n-2,$$

in addition to the condition $R_{g_0} \in L^1$, then we can show that

$$\frac{d}{dt} \int_M |R|^p dvol_{g(t)} \leq 0$$

for $1 \leq p \leq \frac{n}{2}$ under the Yamabe flow (1.3) on asymptotically flat manifolds.

Theorem 3.6. *Let $g(t)$, $0 \leq t \leq T$, be the fine solution to the Yamabe flow (1.2) on an n -dimensional asymptotically flat manifold of order $\tau > 0$ with bounded curvature. Assume that (3.6) holds, $R_{g_0} \geq 0$ and $R_{g_0} \in L^1$. Then*

$$\frac{d}{dt} \int_M |R|^p dvol_{g(t)} \leq 0$$

for $1 \leq p \leq \frac{n}{2}$.

Proof. Let $p = 1 + \epsilon$, where ϵ is sufficient small. Let ϕ be a non-negative cut-off function such that $0 \leq \phi \leq 1$ on M , $\phi = 1$ on $B(o, r)$, $\phi = 0$ outside $B(o, 2r)$ and $|\nabla \phi|^2 \leq 4\frac{\phi}{r^2}$. By Theorem 5.2 in Section 5, we have that $R(g(t)) = O(r^{-\sigma})$, where $\sigma \geq n - 2$. Then

$$\begin{aligned}
& \frac{d}{dt} \int_M \phi^2 |R|^p d\text{vol}_{g(t)} \\
&= \int_M \phi^2 \frac{\partial}{\partial t} (|R|^p u^{\frac{2n}{n-2}}(t)) d\text{vol}_{g(0)} \\
&= \int_M \phi^2 \frac{\partial}{\partial t} (|R|^p) d\text{vol}_{g(t)} + \frac{2n}{n-2} \int_M \phi^2 |R|^p \frac{u_t}{u} d\text{vol}_{g(t)} \\
&= p \int_M \phi^2 |R|^{p-1} \text{sgn}(R) R_t d\text{vol}_{g(t)} - \frac{n}{2} \int_M \phi^2 |R|^p R d\text{vol}_{g(t)} \\
&= p(n-1) \int_M \phi^2 |R|^{p-1} \text{sgn}(R) \Delta R d\text{vol}_{g(t)} + (p - \frac{n}{2}) \int_M \phi^2 |R|^p R d\text{vol}_{g(t)} \\
&= -2p(n-1) \int_M \phi |R|^{p-1} \langle \nabla \phi, \nabla R \rangle \text{sgn}(R) d\text{vol}_{g(t)} \\
&\quad - (n-1)p(p-1) \int_M \phi^2 |R|^{p-2} |\nabla R|^2 d\text{vol}_{g(t)} + (p - \frac{n}{2}) \int_M \phi^2 |R|^p R d\text{vol}_{g(t)} \\
&\leq \frac{2(n-1)p}{p-1} \int_M |\nabla \phi|^2 |R|^p d\text{vol}_{g(t)} - \frac{2(n-1)(p-1)}{p} \int_M \phi^2 |\nabla(|R|^{\frac{p}{2}})|^2 d\text{vol}_{g(t)} \\
&\quad + (p - \frac{n}{2}) \int_M \phi^2 |R|^p R d\text{vol}_{g(t)} \\
&\leq \frac{2(n-1)pCr^{n-2-p\sigma}}{p-1} - \frac{2(n-1)(p-1)}{p} \int_M \phi^2 |\nabla(|R|^{\frac{p}{2}})|^2 d\text{vol}_{g(t)} \\
(3.7) \quad & + (p - \frac{n}{2}) \int_M \phi^2 |R|^p R d\text{vol}_{g(t)}.
\end{aligned}$$

Then we have

$$\frac{d}{dt} \int_M \phi^2 |R|^p d\text{vol}_{g(t)} \leq \frac{2(n-1)pCr^{n-2-p\sigma}}{p-1},$$

for $1 < p \leq \frac{n}{2}$. Then by letting $r \rightarrow \infty$, we have

$$\frac{d}{dt} \int_M |R|^p d\text{vol}_{g(t)} \leq 0.$$

for $1 < p \leq \frac{n}{2}$. By letting $p \rightarrow 1$, we get

$$\frac{d}{dt} \int_M |R| d\text{vol}_{g(t)} \leq 0.$$

□

Then the proof Theorem 1.3 is complete by combining Theorem 3.2, Theorem 3.4 and Theorem 3.5. Finally, we give the proof of Theorem 1.4.

Proof of Theorem 1.4: Since $\frac{\partial}{\partial t}(*dx_i) = O(r^{-\tau-2})$ and $R(g(t)) = O(r^{-\tau-2})$ for $\tau > \frac{n-2}{2}$, we have

$$\begin{aligned}
\frac{d}{dt}m(g(t)) &= \lim_{r \rightarrow \infty} \frac{1}{4\omega_n} \int_{S_r} ((Rg_{ij})_{,j} - (Rg_{jj})_{,i}) * dx_i \\
&\quad + \lim_{r \rightarrow \infty} \frac{1}{4\omega_n} \int_{S_r} (g_{ij,j} - g_{jj,i}) \frac{\partial}{\partial t}(*dx_i) \\
&= \lim_{r \rightarrow \infty} \frac{1}{4\omega_n} \int_{S_r} R(g_{ij,j} - g_{jj,i}) * dx_i \\
&\quad + \lim_{r \rightarrow \infty} \frac{1}{4\omega_n} \int_{S_r} (R_{,j}g_{ij} - R_{,i}g_{jj}) * dx_i \\
&= \lim_{r \rightarrow \infty} \frac{1-n}{4\omega_n} \int_{S_r} R_{,i} * dx_i + \lim_{r \rightarrow \infty} \frac{1}{4\omega_n} \int_{S_r} R_{,i} O(r^{-\tau}) * dx_i \\
&= \lim_{r \rightarrow \infty} \frac{1-n}{4\omega_n} \int_{S_r} R_{,i} * dx_i.
\end{aligned}$$

Note that $g(t)$ satisfies the asymptotic conditions (1.5) of order $\tau > \frac{n-2}{2}$ for $0 \leq t < t_{max}$ by Theorem 3.3. Then $Rc(g(t)) = O(r^{-\tau-2})$ for $\tau > \frac{n-2}{2}$. It follows from Theorem 2.5 that $|\nabla R| = O(r^{-\tau-2})$ for $\tau > \frac{n-2}{2}$. Then we have $\frac{d}{dt}m(g(t)) = 0$ when the dimension $n = 3$ or $n = 4$.

□

4. WEIGHTED SPACES ON ASYMPTOTICALLY FLAT MANIFOLDS

In this section, we study the properties of weighted spaces for parabolic operators defined in Definition 4.1, which will be used to prove Theorem 3.3 in next section.

Similar to Definition 3.1, we can also define the weighted spaces for parabolic operators on asymptotically flat manifolds.

Definition 4.1. Suppose (M^n, g) is an n -dimensional asymptotically flat manifold with asymptotic coordinates $\{x^i\}$. Denote $D_x^j v = \sup_{|\alpha|=j} |\frac{\partial^{|\alpha|}}{\partial x_{i_1} \dots \partial x_{i_j}} v|$.

Let $r(x) = |x|$ on M_∞ and extend r to a smooth positive function on all of M . For parabolic domain $Q_T = M \times [0, T]$, $q \geq 1$ and $\beta \in \mathbb{R}$, the weighted Lebesgue space $L_\beta^q(Q_T)$ is defined as the set of locally

integrable functions v for which the norm

$$\|v\|_{L_\beta^q(Q_T)} = \begin{cases} (\int_0^T \int_M |v|^q r^{-\beta q - n} dx dt)^{\frac{1}{q}}, & q < \infty; \\ \operatorname{ess\,sup}_{Q_T} (r^{-\beta} |v|), & q = \infty. \end{cases}$$

is finite. For an nonnegative even integer k , the weighted Sobolev space $W_\beta^{k,k/2,q}(Q_T)$ is defined as the set of functions v for which the norm

$$\|v\|_{W_\beta^{k,k/2,q}(Q_T)} = \sum_{i+2j \leq k} \|D_x^i D_t^j v\|_{L_{\beta-i-2j}^q(Q_T)}.$$

is finite. Moreover, we also define weighted Sobolev space $\widetilde{W}_\beta^{k,k/2,q}(Q_T)$ as the set of functions v for which the norm

$$\|v\|_{\widetilde{W}_\beta^{k,k/2,q}(Q_T)} = \sum_{i+2j \leq k} \|D_x^i D_t^j v\|_{L_{\beta-i}^q(Q_T)}.$$

is finite. For a nonnegative integer k , the weighted C^k space $C_\beta^k(Q_T)$ is defined as the set of C^k functions v for which the norm

$$\|v\|_{C_\beta^k(Q_T)} = \sum_{i+2j \leq k} \sup_{Q_T} r^{-\beta+i+2j} |D_x^i D_t^j v|$$

is finite. Moreover, we define

$$\begin{aligned} & [v]_{C_\beta^{k+\alpha}(Q_T)} \\ &= \sum_{i+2j=k} \sup_{(x,t) \neq (y,s) \in Q_T} \min(r(x), r(y))^{-\beta+i+2j+\alpha} \frac{|D_x^i D_t^j v(x,t) - D_x^i D_t^j v(y,s)|}{\delta((x,t), (y,s))^\alpha}, \end{aligned}$$

where $\delta((x,t), (y,s)) = |x-y| + |t-s|^{\frac{1}{2}}$ and

$$\begin{aligned} & < v >_{C_\beta^{k+\alpha}(Q_T)} \\ &= \sum_{i+2j=k-1} \sup_{(x,t) \neq (y,s) \in Q_T} r(x)^{-\beta+i+2j+\alpha+1} \frac{|D_x^i D_t^j v(x,t) - D_x^i D_t^j v(y,s)|}{|t-s|^{\frac{\alpha+1}{2}}} \end{aligned}$$

for $k \geq 1$. Then the weighted Hölder space $C_\beta^{k+\alpha, (k+\alpha)/2}(Q_T)$ is defined as the set of functions v for which the norm

$$\|v\|_{C_\beta^{k+\alpha, (k+\alpha)/2}(Q_T)} = \|v\|_{C_\beta^k(Q_T)} + [v]_{C_\beta^{k+\alpha}(Q_T)} + < v >_{C_\beta^{k+\alpha}(Q_T)}$$

is finite. We also use the following weighted Hölder space. The weighted \widetilde{C}^k space $\widetilde{C}_\beta^k(Q_T)$ is defined as the set of C^k functions v for which the norm

$$\|v\|_{\widetilde{C}_\beta^k(Q_T)} = \sum_{i+2j \leq k} \sup_{Q_T} r^{-\beta+i} |D_x^i D_t^j v|$$

is finite. Moreover, we define

$$[v]_{\tilde{C}_\beta^{k+\alpha}(Q_T)} = \sum_{i+2j=k} \sup_{(x,t) \neq (y,t) \in Q_T} \min(r(x), r(y))^{-\beta+i+\alpha} \frac{|D_x^i D_t^j v(x, t) - D_x^i D_t^j v(y, t)|}{|x - y|^\alpha},$$

and

$$\begin{aligned} & \langle v \rangle_{\tilde{C}_\beta^{k+\alpha}(Q_T)} \\ &= \sum_{i+2j=k-1} \sup_{(x,t) \neq (x,s) \in Q_T} r(x)^{-\beta+i} \frac{|D_x^i D_t^j v(x, t) - D_x^i D_t^j v(x, s)|}{|t - s|^{\frac{\alpha+1}{2}}} \end{aligned}$$

where $k \geq 1$. Then the weighted Hölder space $\tilde{C}_\beta^{k+\alpha, (k+\alpha)/2}(Q_T)$ is defined as the set of functions v for which the norm

$$\|v\|_{\tilde{C}_\beta^{k+\alpha, (k+\alpha)/2}(Q_T)} = \|v\|_{\tilde{C}_\beta^k(Q_T)} + [v]_{\tilde{C}_\beta^{k+\alpha}(Q_T)} + \langle v \rangle_{\tilde{C}_\beta^{k+\alpha}(Q_T)}$$

is finite.

Remark 4.2. (i) By Theorem 4.3 (i) below, we see that $\|v\|_{\tilde{W}_\beta^{k, k/2, q}(Q_T)} \leq C \|v\|_{W_\beta^{k, k/2, q}(Q_T)}$.

(ii) For bounded smooth domain $\Omega \subset M^n$, the semi-norm of usual Hölder space is defined as

$$[v]_{C^{k+\alpha}(\Omega_T)} = \sum_{i+2j=k} \sup_{(x,t) \neq (y,s) \in \Omega_T} \frac{|D_x^i D_t^j v(x, t) - D_x^i D_t^j v(y, s)|}{\delta((x, t), (y, s))^\alpha},$$

where $\delta((x, t), (y, s)) = |x - y| + |t - s|^{\frac{1}{2}}$. Note that we do not use parabolic distance in the definition of $[v]_{\tilde{C}_\beta^{k+\alpha}(Q_T)}$ as usual Hölder space on bounded domain. It is because that we want rescaling property (4.7) holds in the proof of Theorem 4.3. In fact, $[v]_{\tilde{C}_\beta^{k+\alpha}(\Omega_T)} \leq C[v]_{C^{k+\alpha}(\Omega_T)}$ for any bounded smooth domain $\Omega \subset M^n$. Clearly, we have $v(x, t) \in C_\beta^{k+\alpha}(M)$ for any $t \in [0, T]$ if $v(x, t) \in \tilde{C}_\beta^{k+\alpha, (k+\alpha)/2}(Q_T)$.

We also have following inequalities which related to the weighted spaces defined in Definition 4.1.

Theorem 4.3. *Suppose (M^n, g) is an n -dimensional asymptotically flat manifold with asymptotic coordinates $\{x^i\}$. Set $Q_T = M^n \times [0, T]$. Then the following inequalities hold:*

(i) For $1 \leq p \leq q \leq \infty$, $\beta_2 < \beta_1$, we have

$$(4.1) \quad \|v\|_{L_{\beta_1}^p(Q_T)} \leq C \|v\|_{L_{\beta_2}^q(Q_T)}.$$

(ii) For $\beta = \beta_1 + \beta_2$, $1 \leq p, q, s \leq \infty$, $\frac{1}{p} = \frac{1}{q} + \frac{1}{s}$, we have

$$(4.2) \quad \|v\|_{L_\beta^p(Q_T)} \leq \|v\|_{L_{\beta_1}^q(Q_T)} \|v\|_{L_{\beta_2}^s(Q_T)},$$

and

$$(4.3) \quad \|v\|_{C_\beta^{\alpha, \alpha/2}(Q_T)} \leq \|v\|_{C_{\beta_1}^{\alpha, \alpha/2}(Q_T)} \|v\|_{C_{\beta_2}^{\alpha, \alpha/2}(Q_T)},$$

here $\|v\|_{C_\beta^{\alpha, \alpha/2}(Q_T)} = \|v\|_{C_\beta^0(Q_T)} + [v]_{C_\beta^\alpha(Q_T)}$ as we defined in Definition 4.1.

(iii) (Sobolev inequalities) For $q \geq 1$, $n \geq 2$, we have

$$(4.4) \quad \|D_x v\|_{L_{\beta-1}^{\frac{(n+2)p}{n+2-p}}(Q_T)} \leq C \|v\|_{\widetilde{W}_\beta^{2,1,q}(Q_T)}$$

if $p < n+2$ and $p \leq q \leq \frac{(n+2)p}{n+2-p}$,

$$(4.5) \quad \|v\|_{L_\beta^{\frac{(n+2)p}{n+2-2p}}(Q_T)} \leq C \|v\|_{\widetilde{W}_\beta^{2,1,q}(Q_T)}$$

if $p < \frac{n+2}{2}$ and $p \leq q \leq \frac{(n+2)p}{n+2-2p}$, and

$$(4.6) \quad \|v\|_{\widetilde{C}_\beta^{m, m/2}(Q_T)} \leq C \|v\|_{\widetilde{W}_\beta^{2,1,p}(Q_T)}$$

if $p > n+2$, $m = 2 - \frac{n+2}{p}$.

Proof. (i) and (4.2) follow from Theorem 1.2 in [5] directly. (4.3) follows from $\|uv\|_{C_\beta^0(Q_T)} \leq \|u\|_{C_{\beta_1}^0(Q_T)} \|v\|_{C_{\beta_2}^0(Q_T)}$ and $[uv]_{C_\beta^\alpha(Q_T)} \leq \|u\|_{C_{\beta_1}^0(Q_T)} [v]_{C_{\beta_2}^\alpha(Q_T)} + [u]_{C_{\beta_1}^\alpha(Q_T)} \|v\|_{C_{\beta_2}^0(Q_T)}$. In order to prove (iii), we use the technique of rescaling, which is used to prove similar results in [5]. Recall a Riemannian manifold M^n with C^∞ metric g is called asymptotically flat of order $\tau > 0$ if there exists a decomposition $M^n = M_0 \cup M_\infty$ with M_0 compact and a diffeomorphism $\phi : M_\infty \rightarrow \mathbb{R}^n - B(o, R_0)$ for some $R_0 > 0$ satisfying (1.5). We denote $A_R = B(o, 2R) \setminus B(o, R)$ be the annulus on \mathbb{R}^n and $E_R = \mathbb{R}^n \setminus B(o, R)$. We consider the function $\phi_*(v|_{M_\infty})$ on \mathbb{R}^n and still denote it by $v|_{M_\infty}$ for simplicity. Consider the rescaled function

$$v_R(x, t) = v(Rx, t).$$

Let $y = Rx$. Then we have $\|D_x^j v_R\|_{L_{\beta-j}^p(A_1 \times [0, T])} = R^\beta \|D_x^j v\|_{L_{\beta-j}^p(A_R \times [0, T])}$ and $\|D_t v_R\|_{L_\beta^p(A_1 \times [0, T])} = R^\beta \|D_t v\|_{L_\beta^p(A_R \times [0, T])}$. It follows that

$$\|v_R(x, t)\|_{\widetilde{W}_\beta^{2,1,p}(A_1 \times [0, T])} = R^\beta \|v(y, t)\|_{\widetilde{W}_\beta^{2,1,p}(A_R \times [0, T])},$$

Set $p^* = \frac{(n+2)p}{n+2-p}$. Note that the weighted Sobolev spaces defined in Definition 4.1 are equivalent to the usual Sobolev spaces on bounded domains. Then we see that

$$\begin{aligned} \|D_x v\|_{L_{\beta-1}^{p^*}(A_R \times [0, T])} &= R^{-\beta} \|D_x v_R\|_{L_{\beta-1}^{p^*}(A_1 \times [0, T])} \\ &\leq C R^{-\beta} \|v_R(x, t)\|_{\widetilde{W}_{\beta}^{2,1,q}(A_1 \times [0, T])} \\ &= C \|v(x, t)\|_{\widetilde{W}_{\beta}^{2,1,q}(A_R \times [0, T])}, \end{aligned}$$

by using the usual Sobolev inequality to A_1 ([22], p.80, Lemma 3.3 or [10], p.29, Theorem 2.3), where C only depends on n, p, A_1 and T^{-1} .

Now we write $v = \sum_{j=0}^{\infty} v_j$ with $v_0 = v|_{M_0}$ and $v_j = v|_{A_{2^{j-1}R_0}}$ for $j \geq 1$, then

$$\begin{aligned} \|D_x v\|_{L_{\beta-1}^{p^*}(Q_T)} &= (\|D_x v_0\|_{L_{\beta-1}^{p^*}(M_0 \times [0, T])}^{p^*} + \sum_{j=1}^{\infty} \|D_x v_j\|_{L_{\beta-1}^{p^*}(A_{2^{j-1}R_0} \times [0, T])}^{p^*})^{\frac{1}{p^*}} \\ &\leq C (\|v_0\|_{\widetilde{W}_{\beta}^{2,1,q}(M_0 \times [0, T])}^{p^*} + \sum_{j=1}^{\infty} \|v_j\|_{\widetilde{W}_{\beta}^{2,1,q}(A_{2^{j-1}R_0} \times [0, T])}^{p^*})^{\frac{1}{p^*}} \\ &\leq C (\|v_0\|_{\widetilde{W}_{\beta}^{2,1,q}(M_0 \times [0, T])}^q + \sum_{j=1}^{\infty} \|v_j\|_{\widetilde{W}_{\beta}^{2,1,q}(A_{2^{j-1}R_0} \times [0, T])}^q)^{\frac{1}{q}}, \end{aligned}$$

since $p^* \geq q$ and $(\sum a_j^{p^*})^{\frac{1}{p^*}} \leq (\sum a_j^q)^{\frac{1}{q}}$ for $a_j \geq 0$. Therefore, (4.4) holds clearly. Since we have

$$(4.7) \quad \|v_R\|_{\widetilde{C}_{\beta}^{k+\alpha, (k+\alpha)/2}(A_1 \times [0, T])} = R^{\beta} \|v\|_{\widetilde{C}_{\beta}^{k+\alpha, (k+\alpha)/2}(A_R \times [0, T])},$$

(4.5) and (4.6) follow from same rescaling arguments and usual Sobolev inequalities ([22], p.80, Lemma 3.3 or [10], p.29, Theorem 2.3 and p.38, Theorem 3.4). \square

The advantage of weighted Sobolev spaces is that they give us analogues of some global elliptic or parabolic regularity results for compact manifolds. We have the following weighted estimates hold.

Theorem 4.4. *Suppose (M^n, g_0) is an n -dimensional asymptotically flat manifold of order $\tau > 0$, $p > 1$. Then there is a constant $C = C(n, p, \tau, \beta)$ such that*

$$(4.8) \quad \|v\|_{W_{\beta}^{2,1,p}(Q_T)} \leq C (\|(\partial_t - \Delta_{g_0})v\|_{L_{\beta-2}^p(Q_T)} + \|v\|_{L_{\beta}^p(Q_T)}),$$

where $v(x, 0) = 0$.

Proof. We use the same notations as the proof of Theorem 4.3. Let $A_R = B(o, 2R) \setminus B(o, R)$ and $E_R = \mathbb{R}^n \setminus B(o, R)$. Denote Δ_0 be the stand Laplacian on E_{R_0} with flat metric. Here we consider the rescaled function

$$v_R(x, t) = v(Rx, R^2t).$$

Let $y = Rx$, $\bar{t} = R^2t$. By a simple change of variables, we have $\|D_x^j v_R\|_{L_{\beta-j}^p(A_1 \times [0, T])} = R^{\beta - \frac{2}{p}} \|D_x^j v\|_{L_{\beta-j}^p(A_R \times [0, R^2T])}$ and $\|\frac{\partial}{\partial t} v_R\|_{L_{\beta-2}^p(A_1 \times [0, T])} = R^{\beta - \frac{2}{p}} \|\frac{\partial}{\partial \bar{t}} v\|_{L_{\beta-2}^p(A_R \times [0, R^2T])}$. Hence,

$$\|v_R\|_{W_{\beta}^{2,1,p}(A_1 \times [0, T])} = R^{\beta - \frac{2}{p}} \|v\|_{W_{\beta}^{2,1,p}(A_R \times [0, R^2T])},$$

and

$$\|(\frac{\partial}{\partial t} - \Delta_0)v_R\|_{L_{\beta-2}^p(A_1 \times [0, T])} = R^{\beta - \frac{2}{p}} \|(\frac{\partial}{\partial \bar{t}} - \Delta_0)v\|_{L_{\beta-2}^p(A_R \times [0, R^2T])}.$$

We write $v = \sum_{j=0}^{\infty} v_j$ with $v_0 = v|_{M_0}$ and $v_j = v|_{A_{2^{j-1}R}}$ for $j \geq 1$, where $R \geq R_0$. Note that v_j vanishes outside $A_{2^{j-1}R}$. By the standard L^p estimates for parabolic equation,

$$\begin{aligned} \|v_j\|_{W_{\beta}^{2,1,p}(A_{2^{j-1}R} \times [0, T])} &= (2^{j-1}R)^{-(\beta - \frac{2}{p})} \| (v_j)_R \|_{W_{\beta}^{2,1,p}(A_1 \times [0, (2^{j-1}R)^{-2}T])} \\ &\leq C(2^{j-1}R)^{-(\beta - \frac{2}{p})} (\|(\partial_t - \Delta_0)(v_j)_R\|_{L_{\beta-2}^p(A_1 \times [0, (2^{j-1}R)^{-2}T])} + \|(v_j)_R\|_{L_{\beta}^p(A_1 \times [0, (2^{j-1}R)^{-2}T])}) \\ &= C(\|(\partial_t - \Delta_0)v_j\|_{L_{\beta-2}^p(A_{2^{j-1}R} \times [0, T])} + \|v_j\|_{L_{\beta}^p(A_{2^{j-1}R} \times [0, T])}), \end{aligned}$$

where C is independent of j , R and T (see Proposition 7.11 in [23]). Therefore,

$$\begin{aligned} \|v\|_{W_{\beta}^{2,1,p}(E_R \times [0, T])} &= \left(\sum_{j=1}^{\infty} \|v_j\|_{W_{\beta}^{2,1,p}(A_{2^{j-1}R} \times [0, T])}^p \right)^{\frac{1}{p}} \\ &\leq C \left(\sum_{j=1}^{\infty} (\|(\partial_t - \Delta_0)v_j\|_{L_{\beta-2}^p(A_{2^{j-1}R} \times [0, T])} + \|v_j\|_{L_{\beta}^p(A_{2^{j-1}R} \times [0, T])})^p \right)^{\frac{1}{p}} \\ (4.9) \quad &\leq C(\|(\partial_t - \Delta_0)v\|_{L_{\beta-2}^p(E_R \times [0, T])} + \|v\|_{L_{\beta}^p(E_R \times [0, T])}). \end{aligned}$$

Since $\Delta_{g_0} = \frac{1}{\sqrt{\det g_0}} \frac{\partial}{\partial x_i} (\sqrt{\det g_0} g_0^{ij} \frac{\partial}{\partial x^j})$, we write $\Delta_{g_0} = g_0^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + b^j \frac{\partial}{\partial x^j}$, where $b^j = \frac{1}{\sqrt{\det g_0}} \frac{\partial}{\partial x_i} (\sqrt{\det g_0} g_0^{ij})$. Then if $\text{supp}(v) \subset E_R$ and $q > n + 2$,

we compute

$$\begin{aligned}
& \|(\Delta_0 - \Delta_{g_0})v\|_{L_{\beta-2}^p(Q_T)} \\
& \leq \sup_{|x|>R} |g_0^{ij} - \delta_{ij}| \|D_x^2 v\|_{L_{\beta-2}^p(Q_T)} + \|b\|_{L_{-1}^q(E_R \times [0, T])} \|D_x v\|_{L_{\beta-1}^{\frac{pq}{q-p}}(Q_T)} \\
& \leq \sup_{|x|>R} |g_0^{ij} - \delta_{ij}| \|D_x^2 v\|_{L_{\beta-2}^p(Q_T)} + C \|b\|_{L_{-1}^q(E_R \times [0, T])} \|v\|_{W_{\beta}^{2,1,p}(Q_T)} \\
(4.10) \quad & \leq \left(\sup_{|x|>R} |g_0^{ij} - \delta_{ij}| + C \|b\|_{L_{-1}^q(E_R \times [0, T])} \right) \|v\|_{W_{\beta}^{2,1,p}(Q_T)},
\end{aligned}$$

by using Theorem 4.3. Let $\zeta \in C_0^\infty(B_2)$ such that $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ in B_1 . Set $\zeta_R(x) = \zeta(x/R)$. Writing $v = v_0 + v_\infty$, $v_0 = \zeta_R v$, $v_\infty = (1 - \zeta_R)v$, where R is a constant to be determined. Then (4.9) yields

$$\begin{aligned}
\|v_\infty\|_{W_{\beta}^{2,1,p}(Q_T)} & \leq C(\|(\partial_t - \Delta_0)v_\infty\|_{L_{\beta-2}^p(Q_T)} + \|v_\infty\|_{L_{\beta}^p(Q_T)}) \\
& \leq C(\|(\partial_t - \Delta_{g_0})v_\infty\|_{L_{\beta-2}^p(Q_T)} + \|(\Delta_{g_0} - \Delta_0)v_\infty\|_{L_{\beta-2}^p(Q_T)} \\
(4.11) \quad & + \|v_\infty\|_{L_{\beta}^p(Q_T)})
\end{aligned}$$

Moreover, by the asymptotic condition (1.5) and Theorem 4.3 (i),

$$(4.12) \quad \sup_{|x|>R} |g_0^{ij} - \delta_{ij}| + C \|b\|_{L_{-1}^q(E_R \times [0, T])} \rightarrow 0,$$

as $R \rightarrow \infty$. Using (4.10), (4.11) and (4.12), we obtain

$$(4.13) \quad \|v_\infty\|_{W_{\beta}^{2,1,p}(Q_T)} \leq C(\|(\partial_t - \Delta_{g_0})v_\infty\|_{L_{\beta-2}^p(Q_T)} + \|v_\infty\|_{L_{\beta}^p(Q_T)})$$

for R sufficient large. Then we estimate

$$\begin{aligned}
& \|(\partial_t - \Delta_{g_0})v_\infty\|_{L_{\beta-2}^p(Q_T)} \\
& \leq \|(\partial_t - \Delta_{g_0})v\|_{L_{\beta-2}^p(Q_T)} + \|(\partial_t - \Delta_{g_0})(\zeta_R v)\|_{L_{\beta-2}^p(Q_T)} \\
& \leq 2\|(\partial_t - \Delta_{g_0})v\|_{L_{\beta-2}^p(Q_T)} + \|v \Delta_{g_0} \zeta_R + 2\nabla_{g_0} u \cdot \nabla_{g_0} \zeta_R\|_{L_{\beta-2}^p(A_R \times [0, T])} \\
(4.14) \quad & \leq 2\|(\partial_t - \Delta_{g_0})v\|_{L_{\beta-2}^p(Q_T)} + C\|v + |\nabla_{g_0} u|\|_{L^p(A_R \times [0, T])}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \|(\partial_t - \Delta_{g_0})v_0\|_{L_{\beta-2}^p(Q_T)} \\
(4.15) \quad & \leq \|(\partial_t - \Delta_{g_0})v\|_{L_{\beta-2}^p(Q_T)} + C\|v + |\nabla_{g_0} u|\|_{L^p(A_R \times [0, T])}.
\end{aligned}$$

By using the interpolation inequality, standard parabolic L^p estimate on v_0 , (4.13), (4.14) and (4.15), we conclude that (4.8) holds. \square

Theorem 4.5. *Suppose (M^n, g_0) is an n -dimensional asymptotically flat manifold of order $\tau > 0$. Then there is a constant $C = C(n, p, \tau, \beta)$ such that*

$$(4.16) \quad \|v\|_{C_{\beta}^{2+\alpha, 1+\alpha/2}(Q_T)} \leq C(\|(\partial_t - \Delta_{g_0})v\|_{C_{\beta}^{\alpha, \alpha/2}(Q_T)} + \|v\|_{C_{\beta}^0(Q_T)}),$$

where $v(x, 0) = 0$.

Proof. We use the same notations as the proof of Theorem 4.3. We also consider the rescaled function

$$v_R(x, t) = v(Rx, R^2t).$$

Let $y = Rx$, $\bar{t} = R^2t$. We have $\|v_R\|_{C_{\beta}^{k+\alpha, (k+\alpha)/2}(A_1 \times [0, T])} = R^{\beta} \|v\|_{C_{\beta}^{k+\alpha, (k+\alpha)/2}(A_R \times [0, R^2T])}$ for $R \geq R_0$. Using the similar scaling arguments to Theorem 4.4 and the standard Schauder estimate for parabolic equations (see Theorem 4.9 in [23]), we can obtain

$$(4.17) \quad \|v\|_{C_{\beta}^{2+\alpha, 1+\alpha/2}(E_R \times [0, T])} \leq C(\|(\partial_t - \Delta_0)v\|_{C_{\beta-2}^{\alpha, \alpha/2}(E_R \times [0, T])} + \|v\|_{C_{\beta}^0(E_R \times [0, T])}).$$

If $\text{supp}(v) \subset E_R$, we compute

$$(4.18) \quad \begin{aligned} & \|(\Delta_0 - \Delta_{g_0})v\|_{C_{\beta-2}^{\alpha, \alpha/2}(Q_T)} \\ & \leq \|g_0^{ij} - \delta_{ij}\|_{C_0^{\alpha, \alpha/2}(E_R \times [0, T])} \|D_x^2 v\|_{C_{\beta-2}^{\alpha, \alpha/2}(Q_T)} + \|b\|_{C_{-1}^{\alpha, \alpha/2}(E_R \times [0, T])} \|D_x v\|_{C_{\beta-1}^{\alpha, \alpha/2}(Q_T)} \\ & \leq (\|g_0^{ij} - \delta_{ij}\|_{C_0^{\alpha, \alpha/2}(E_R \times [0, T])} + \|b\|_{C_{-1}^{\alpha, \alpha/2}(E_R \times [0, T])}) \|v\|_{C_{\beta}^{2+\alpha, 1+\alpha/2}(Q_T)}, \end{aligned}$$

by using Theorem 4.3. Note that

$$(4.19) \quad \|g_0^{ij} - \delta_{ij}\|_{C_0^{\alpha, \alpha/2}(E_R \times [0, T])} + \|b\|_{C_{-1}^{\alpha, \alpha/2}(E_R \times [0, T])} \rightarrow 0,$$

as $R \rightarrow \infty$ by the asymptotic condition (1.5). Set $\zeta_R(x) = \zeta(x/R)$. We writing $v = v_0 + v_{\infty}$, $v_0 = \zeta_R v$, $v_{\infty} = (1 - \zeta_R)v$, where R is a constant to be determined. Then Theorem 4.5 holds by estimating v_0 and v_{∞} as the similar way to Theorem 4.4. \square

5. WEIGHTED ESTIMATES FOR YAMABE FLOW

In this section, we give the proof of Theorem 3.3. The key part is to prove $1 - u(x, t) \in C_{-\tau}^{2+\alpha, 1+\alpha/2}(Q_T)$ (this implies $g_{ij}(x, t) - \delta_{ij} \in C_{-\tau}^{2+\alpha}(M)$ for $0 \leq t < t_{max}$), where $u(x, t)$ is the fine solution to the Yamabe flow. We solve this problem by the global L^p estimate and Schauder estimate for the weighted spaces (see Theorem 4.3, Theorem 5.3 and Theorem 5.5). Set $v(x, t) = 1 - u(x, t)$, where $u(x, t)$ is the

fine solution to Yamabe flow (1.3). We first show $v(x, t)$ belongs to weighted space $C_{-(\tau+2)}^0(Q_T)$.

Theorem 5.1. *Let $u(x, t)$, $0 \leq t \leq T$, be the fine solution to the Yamabe flow (1.3) on an n -dimensional asymptotically flat manifold (M^n, g_0) of order $\tau > 0$. Set $v = 1 - u$. Then*

$$(5.1) \quad v(x, t) = O(r^{-(\tau+2)}).$$

for all $t \in [0, T]$ and $v(x, t) \in C_{-(\tau+2)}^0(Q_T)$.

Proof. Setting $v = 1 - u$, we have

$$(5.2) \quad N(1 - v)^{N-1}v_t = \Delta_{g_0}v - aR_{g_0}v + aR_{g_0}.$$

Let $f(x)$ be the function defined in Lemma 2.7. Set $h(x) = f(x)^{\tau+2}$ and $w = h(x)v(x, t)$. Then, by direct computation we have

$$\begin{aligned} N(1 - v)^{N-1}w_t &= \Delta_{g_0}w - 2\nabla_{g_0} \log h \cdot \nabla_{g_0}w \\ &\quad - w\left(\frac{\Delta_{g_0}h}{h} - 2\frac{|\nabla_{g_0}h|^2}{h^2} + aR_{g_0}\right) + ahR_{g_0}. \end{aligned}$$

Hence,

$$\begin{aligned} w_t &= \operatorname{div}_{g_0}\left(\frac{1}{N(1 - v)^{N-1}}\nabla_{g_0}w\right) + b \cdot \nabla_{g_0}w \\ &\quad + dw + \frac{a}{N(1 - v)^{N-1}}hR_{g_0}, \end{aligned}$$

where $b(w, t) = -(\nabla_{g_0}(\frac{1}{N(1-v)^{N-1}}) + 2\nabla_{g_0} \log h)$ and $d(x, t) = -\frac{1}{N(1-v)^{N-1}}(\frac{\Delta_{g_0}h}{h} - 2\frac{|\nabla_{g_0}h|^2}{h^2} + aR_{g_0})$. Note that $R_{g_0} = \partial_j(\partial_i g_{ij} - \partial_j g_{ii}) + O(r^{-2\tau-2}) = O(r^{-(\tau+2)})$. Therefore, $|\frac{a}{N(1-v)^{N-1}}hR_{g_0}| \leq C$. Moreover, $|d| \leq D$ by Lemma 2.7 and the assumptions of Theorem 5.1. Set $\tilde{w} = w - e^{(D+DCT)t} - Ct$. Then we have

$$\begin{aligned} \tilde{w}_t &\leq \operatorname{div}_{g_0}\left(\frac{1}{N(1 - v)^{N-1}}\nabla_{g_0}\tilde{w}\right) + b \cdot \nabla_{g_0}\tilde{w} \\ &\quad + d\tilde{w} - De^{(D+DCT)t} - DCTe^{(D+DCT)t} \\ &\leq \operatorname{div}_{g_0}\left(\frac{1}{N(1 - v)^{N-1}}\nabla_{g_0}\tilde{w}\right) + b \cdot \nabla_{g_0}\tilde{w} + d\tilde{w}. \end{aligned}$$

Then (5.1) follows from Theorem 2.4 immediately. \square

We first need the following lemma.

Lemma 5.2. *Let $u(x, t)$, $0 \leq t < t_{max}$, be the fine solution to the Yamabe flow (1.3) on an n -dimensional asymptotically flat manifold (M^n, g_0) of order $\tau > 0$. Set $g(t) = u^{\frac{4}{n-2}}g_0$. If the scalar curvature at*

$t = 0$ satisfies the decay condition $R_{g_0} = O(r^{-q})$ for some $q > 0$, then $R(g(t)) = O(r^{-q})$ for $0 \leq t < t_{max}$ and $R(g(t)) \in C_{-(\tau+2)}^0(Q_T)$.

Proof. By a small right shift at time $t = 0$ and using Theorem 2.5, we may assume that $|\nabla R|$ are bounded on $[0, t]$. The scalar curvature of $g(t)$ evolves as

$$\frac{\partial}{\partial t} R = (n-1)\Delta R + R^2.$$

along the Yamabe flow (1.2). Let $w = f^q R = hR$, where f is the function defined in Lemma 2.7. So $w(x, 0) \leq D$. Then we have

$$\left(\frac{\partial}{\partial t} - (n-1)\Delta\right)w = bw - 2(n-1)\nabla \log h \cdot \nabla w,$$

where $b = R - \frac{(n-1)}{h}(\Delta h - \frac{2|\nabla h|^2}{h})$. Note that $|b| \leq B$ by Lemma 2.7. Setting $\tilde{w} = w - De^{tB}$, we obtain

$$\left(\frac{\partial}{\partial t} - (n-1)\Delta\right)\tilde{w} \leq b\tilde{w} - 2(n-1)\nabla \log h \cdot \nabla \tilde{w},$$

Then the maximum principle implies (Theorem 2.4) that $\tilde{w}(t) \leq 0$ if $\tilde{w}(0) \leq 0$. □

Using the similar arguments to Theorem 4.4, we have the following estimate holds.

Theorem 5.3. *Let $u(x, t)$, $0 \leq t \leq T$, be the fine solution to the Yamabe flow (1.3) on an n -dimensional asymptotically flat manifold (M^n, g_0) of order $\tau > 0$. Assume $0 < \delta \leq u(x, t) \leq C'$ on $[0, T]$. Set $v = 1 - u$. Then there is a constant $C = C(n, p, \tau, \beta, \delta, C')$ such that*

$$(5.3) \quad \|v\|_{W_{\beta}^{2,1,p}(Q_T)} \leq C(\|R_{g_0}\|_{L_{\beta-2}^p(Q_T)} + \|v\|_{L_{\beta}^p(Q_T)}).$$

Proof. We use the same definitions as the proof of Theorem 4.3. Recall $v = 1 - u$ evolves as

$$(5.4) \quad N(1-v)^{N-1}v_t = \Delta_{g_0}v - aR_{g_0}v + aR_{g_0}$$

along the Yamabe flow (1.3). We change time by a constant scale such that (5.4) is equivalent to the following equation

$$(5.5) \quad (1-v)^{N-1}v_t = \Delta_{g_0}v - aR_{g_0}v + aR_{g_0}.$$

Set $Pv = h(\Delta_{g_0}v - aR_{g_0}v) = h(g_0^{ij}\frac{\partial^2 v}{\partial x^i \partial x^j} + b^j\frac{\partial v}{\partial x^j} - aR_{g_0}v)$, where $h = \frac{1}{(1-v)^{N-1}}$ and $b^j = \frac{1}{\sqrt{\det g_0}}\frac{\partial}{\partial x_i}(\sqrt{\det g_0}g_0^{ij})$. Denote Δ_0 be the stand

Laplacian on E_{R_0} with flat metric. If $\text{supp}(v) \subset E_R$, then

$$\begin{aligned}
& \|(\Delta_0 - P)v\|_{L_{\beta-2}^p(Q_T)} \\
& \leq \sup_{|x|>R} |hg_0^{ij} - \delta_{ij}| \|D_x^2 v\|_{L_{\beta-2}^p(Q_T)} + \|hb^j \frac{\partial v}{\partial x^j}\|_{L_{\beta-2}^p(Q_T)} \\
& \quad + \|ahR_{g_0}v\|_{L_{\beta-2}^p(Q_T)} \\
& \leq \sup_{|x|>R} |hg_0^{ij} - \delta_{ij}| \|D_x^2 v\|_{L_{\beta-2}^p(Q_T)} + C\|b\|_{L_{-1}^q(E_R \times [0,T])} \|D_x v\|_{L_{\beta-1}^{\frac{pq}{q-p}}(Q_T)} \\
& \quad + C\|R_{g_0}\|_{L_{-2}^{\frac{q}{2}}(E_R \times [0,T])} \|v\|_{L_{\beta}^{\frac{pq}{q-2p}}(Q_T)} \\
& \leq \sup_{|x|>R} |hg_0^{ij} - \delta_{ij}| \|D_x^2 v\|_{L_{\beta-2}^p(Q_T)} + C\|b\|_{L_{-1}^q(E_R \times [0,T])} \|v\|_{W_{\beta}^{2,1,p}(Q_T)} \\
& \quad + C\|R_{g_0}\|_{L_{-2}^{\frac{q}{2}}(E_R \times [0,T])} \|v\|_{W_{\beta}^{2,1,p}(Q_T)} \\
& \leq (\sup_{|x|>R} |hg_0^{ij} - \delta_{ij}| + C\|b\|_{L_{-1}^q(E_R \times [0,T])} + C\|R_{g_0}\|_{L_{-2}^{\frac{q}{2}}(E_R \times [0,T])}) \|v\|_{W_{\beta}^{2,1,p}(Q_T)},
\end{aligned}$$

by using Theorem 4.3. Note that

$$(5.6) \quad \sup_{|x|>R} |hg_0^{ij} - \delta_{ij}| + C\|b\|_{L_{-1}^q(E_R \times [0,T])} + C\|R_{g_0}\|_{L_{-2}^{\frac{q}{2}}(E_R \times [0,T])} \rightarrow 0,$$

as $R \rightarrow \infty$, by the asymptotic condition (1.5) and Theorem 4.3 (i). We already known that

$$\|v\|_{W_{\beta}^{2,1,p}(E_R \times [0,T])} \leq C(\|(\partial_t - \Delta_0)v\|_{L_{\beta-2}^p(E_R \times [0,T])} + \|v\|_{L_{\beta}^p(E_R \times [0,T])})$$

for $R \geq R_0$ by (4.9). So by using the similar arguments of Theorem 4.4, we conclude that

$$\|v\|_{W_{\beta}^{2,1,p}(Q_T)} \leq C(\|(\partial_t - P)v\|_{L_{\beta-2}^p(Q_T)} + \|v\|_{L_{\beta}^p(Q_T)}).$$

Then Theorem 5.3 holds immediately. \square

As an immediate application of Theorem 5.1 and Theorem 5.3, we have the following corollary holds.

Corollary 5.4. *Let $u(x, t)$, $0 \leq t \leq T$, be the fine solution to the Yamabe flow (1.3) on an n -dimensional asymptotically flat manifold (M^n, g_0) of order $\tau > 0$. Set $g(x, t) = u^{\frac{4}{n-2}}(x, t)g_0$. Then $g_{ij}(x, t) - \delta_{ij} \in \tilde{C}_{-\tau'}^{1+\alpha, (1+\alpha)/2}(Q_T)$, where $\tau' = \tau - \epsilon$ for any $\epsilon > 0$.*

Proof. By Theorem 5.1, we have $v \in L_{-\tau-2}^{\infty}(Q_T)$. Then $R_{g_0} \in L_{-\tau-2+\epsilon}^p(Q_T)$ and $v \in L_{-\tau+\epsilon}^p(Q_T)$ for any $1 < p < \infty$, $\epsilon > 0$. It follows from Theorem 5.3 that $v \in W_{-\tau+\epsilon}^{2,1,p}$ for any $1 < p < \infty$, $\epsilon > 0$. Then $v \in \tilde{C}_{-\tau+\epsilon}^{1+\alpha, (1+\alpha)/2}(Q_T)$ by (4.6). \square

We also have the following estimate.

Theorem 5.5. *Let $u(x, t)$, $0 \leq t \leq T$, be the fine solution to the Yamabe flow (1.3) on an n -dimensional asymptotically flat manifold (M^n, g_0) of order $\tau > 0$. Set $v = 1 - u$. Assume $0 < \delta \leq u(x, t) \leq C'$ on $[0, T]$, $\|v\|_{C_0^{\alpha, \alpha/2}(Q_T)} \leq C''$ and $R_{g_0} \in C_{-2-\tau}^\alpha(M)$. Then there is a constant $C = C(n, p, \tau, \beta, \delta, C', C'')$ such that*

$$(5.7) \quad \|v\|_{C_\beta^{2+\alpha, 1+\alpha/2}(Q_T)} \leq C(\|R_{g_0}\|_{C_{\beta-2}^{\alpha, \alpha/2}(Q_T)} + \|v\|_{C_\beta^0(Q_T)}).$$

Proof. We use the same notations as the proof of Theorem 4.4. Since $0 < \delta \leq u(x, t) \leq C'$ on $[0, T]$ and $\|v\|_{C_0^{\alpha, \alpha/2}(Q_T)} \leq C''$, $\|h\|_{C_0^{\alpha, \alpha/2}(Q_T)} \leq C'''$, where C''' is a constant only depending on δ, N, C', C'' . If $\text{supp}(v) \subset E_R$, we compute

$$\begin{aligned} & \|(\Delta_0 - P)v\|_{C_{\beta-2}^{\alpha, \alpha/2}(Q_T)} \\ & \leq \|hg_0^{ij} - \delta_{ij}\|_{C_0^{\alpha, \alpha/2}(E_R \times [0, T])} \|D_x^2 v\|_{C_{\beta-2}^{\alpha, \alpha/2}(Q_T)} \\ & \quad + \|h\|_{C_0^{\alpha, \alpha/2}(E_R \times [0, T])} \|b\|_{C_{-1}^{\alpha, \alpha/2}(E_R \times [0, T])} \|D_x v\|_{C_{\beta-1}^{\alpha, \alpha/2}(Q_T)} \\ & \quad + \|h\|_{C_0^{\alpha, \alpha/2}(E_R \times [0, T])} \|R_{g_0}\|_{C_{-2}^{\alpha, \alpha/2}(E_R \times [0, T])} \|v\|_{C_\beta^{\alpha, \alpha/2}(Q_T)} \\ & \leq C(\|hg_0^{ij} - \delta_{ij}\|_{C_0^{\alpha, \alpha/2}(E_R \times [0, T])} + \|b\|_{C_{-1}^{\alpha, \alpha/2}(E_R \times [0, T])} \\ (5.8) \quad & + \|R_{g_0}\|_{C_{-2}^{\alpha, \alpha/2}(E_R \times [0, T])}) \|v\|_{C_\beta^{2+\alpha, 1+\alpha/2}(Q_T)} \end{aligned}$$

by using Theorem 4.3. Note that

$$(5.9) \quad \|hg_0^{ij} - \delta_{ij}\|_{C_0^{\alpha, \alpha/2}(E_R \times [0, T])} + \|b\|_{C_{-1}^{\alpha, \alpha/2}(E_R \times [0, T])} + \|R_{g_0}\|_{C_{-2}^{\alpha, \alpha/2}(E_R \times [0, T])} \rightarrow 0,$$

as $R \rightarrow \infty$ by the asymptotic condition (1.5), $R_{g_0} \in C_{-2-\tau}^\alpha(M)$, and Theorem 4.3. We already known that

$$\|v\|_{C_\beta^{2+\alpha, 1+\alpha/2}(E_R \times [0, T])} \leq C(\|(\partial_t - \Delta_0)v\|_{C_{\beta-2}^{\alpha, \alpha/2}(E_R \times [0, T])} + \|v\|_{C_\beta^0(E_R \times [0, T])})$$

for $R \geq R_0$ by (4.17). So by using the similar arguments of Theorem 4.3, we see that Theorem 5.5 holds. \square

Now we can give the proof of Theorem 3.3.

Proof of Theorem 3.3. It follows from Corollary 5.4 that $v \in \tilde{C}_{-\tau+\epsilon}^{1+\alpha, (1+\alpha)/2}(Q_T)$ for any $\epsilon > 0$. Then

$$\begin{aligned}
& \|v\|_{C_0^{\alpha, \alpha/2}(Q_T)} \\
&= \|v\|_{C_0^\alpha(Q_T)} + \sup_{(x,t) \neq (y,s) \in Q_T} \min(r(x), r(y))^\alpha \frac{|v(x,t) - v(y,s)|}{\delta((x,t), (y,s))^\alpha} \\
&\leq C(\|v\|_{C_0^\alpha(Q_T)} + \sup_{(x,t) \neq (y,s) \in Q_T} \min(r(x), r(y))^\alpha \frac{|v(x,t) - v(y,t)|}{|x-y|^\alpha} \\
&\quad + \sup_{(x,t) \neq (y,s) \in Q_T} \min(r(x), r(y))^\alpha \frac{|v(y,t) - v(y,s)|}{|t-s|^{\frac{\alpha}{2}}}) \\
&\leq C\|v\|_{\tilde{C}_{-\tau+\epsilon}^{1+\alpha, (1+\alpha)/2}(Q_T)}
\end{aligned}$$

for $\epsilon > 0$ small. Since $R_{g_0} \in C_{-2-\tau}^\alpha(M)$ and R_{g_0} is independent of time, we have $R_{g_0} \in C_{-2-\tau}^{\alpha, \alpha/2}(Q_T)$. By Theorem 5.1, we know that $v \in C_{-2-\tau}^0(Q_T)$. It follows from Theorem 5.5 that $v \in C_{-\tau}^{2+\alpha, 1+\alpha/2}(Q_T)$. It implies $1-u(x,t) \in C_{-\tau}^{2+\alpha}(M)$ for $0 \leq t < t_{max}$ by the Definition 4.1. \square

6. APPENDIX

In this section, we give a proof of Theorem 2.4 for sake of convenience for the readers.

Proof of Theorem 2.4: Define $\theta > 0$ to be chosen

$$h(y, t) = -\frac{\theta d_{g(t)}^2(p, y)}{4(2\eta - t)}, 0 < t < \eta,$$

where $d_{g(t)}(p, y)$ is the distance between p and y at time t and $0 < \eta < \min(T, \frac{1}{64K}, \frac{1}{32\alpha_4}, \frac{1}{4\alpha_5})$. Then

$$\frac{d}{dt}h = -\frac{\theta d_{g(t)}^2(p, y)}{4(2\eta - t)^2} - \frac{\theta d_{g(t)}(p, y)}{2(2\eta - t)} \frac{d}{dt}d_{g(t)}(p, y).$$

By (iv), we have

$$|\frac{d}{dt}d_{g(t)}(p, y)| \leq \frac{1}{2}\alpha_5 d_{g(t)}(p, y).$$

Then we conclude that

$$\frac{d}{dt}h \leq -\theta^{-1}|\nabla h|^2 + \theta^{-1}\alpha_5|\nabla h|^2(2\eta - t),$$

We choose $\theta = \frac{1}{4\alpha_1}$, then

$$(6.1) \quad \frac{d}{dt}h + 2a|\nabla h|^2 \leq 0$$

by using $\eta \leq \frac{1}{4\alpha_5}$. Taking $f_K = \max\{\min(f, K), 0\}$ and $0 < \epsilon < \eta$, we have

$$\begin{aligned} & \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h f_K (\operatorname{div}(a \nabla f) - \frac{\partial f}{\partial t}) d\mu_t \right) dt \\ & \geq -\alpha_2 \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h f_K |\nabla f| d\mu_t \right) dt \\ & \quad - \alpha_3 \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h f_K f d\mu_t \right) dt \end{aligned}$$

for some smooth time independent compactly supported function ϕ on M^n , where $\beta > 0$ will be chosen later. Then we have

$$\begin{aligned} 0 & \leq - \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h a < \nabla f_K, \nabla f_K > d\mu_t \right) dt \\ & \quad - \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h f_K a < \nabla h, \nabla f > d\mu_t \right) dt \\ & \quad - 2 \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi e^h f_K a < \nabla \phi, \nabla f > d\mu_t \right) dt \\ & \quad - \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h f_K \frac{\partial f}{\partial t} d\mu_t \right) dt + \alpha_3 \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h f_K f d\mu_t \right) dt \\ & \quad + \alpha_2 \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h f_K |\nabla f| d\mu_t \right) dt \\ & = \text{I} + \text{II} + \text{III} + \text{IV} + \text{V} + \text{VI}. \end{aligned}$$

By Schwartz' inequality, we derive

$$\text{II} \leq \frac{1}{4} \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h a |\nabla f|^2 d\mu_t \right) dt + \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h f_K^2 a |\nabla h|^2 d\mu_t \right) dt,$$

$$\text{III} \leq \frac{1}{2} \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h a |\nabla f|^2 d\mu_t \right) dt + 2 \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M e^h f_K^2 a |\nabla \phi|^2 d\mu_t \right) dt,$$

and

$$\begin{aligned} \text{VI} & \leq \frac{1}{4} \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h a |\nabla f|^2 d\mu_t \right) dt + \alpha_2^2 \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M e^h f_K^2 \frac{1}{a} |\nabla \phi|^2 d\mu_t \right) dt \\ & \leq \frac{1}{4} \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h a |\nabla f|^2 d\mu_t \right) dt + \frac{\alpha_2^2}{\alpha_1'} \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M e^h f_K^2 |\nabla \phi|^2 d\mu_t \right) dt. \end{aligned}$$

Since

$$-e^h f_K \frac{\partial f}{\partial t} \leq -e^h f_K \frac{\partial f_K}{\partial t} + \frac{\partial}{\partial t} (e^h f_K (f_K - f)),$$

and

$$f_K (f_K - f) \leq 0,$$

we obtain

$$\begin{aligned}
& \text{IV} + \text{V} \\
& \leq -\frac{1}{2} \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h \frac{\partial f_K^2}{\partial t} d\mu_t \right) dt + \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi^2 \frac{\partial}{\partial t} (e^h f_K (f_K - f)) d\mu_t \right) dt \\
& - \alpha_3 \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h f_K (f_K - f) d\mu_t \right) dt + \alpha_3 \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h f_K^2 d\mu_t \right) dt.
\end{aligned}$$

Moreover, we have

$$\left| \frac{d}{dt} (d\mu_t) \right| \leq n\alpha_5 d\mu_t$$

by (iv). Now we choose $\beta \geq 2n\alpha_5 + 4\alpha_3 + 4\frac{\alpha_2^2}{\alpha_1}$. Then

$$\begin{aligned}
& \text{IV} + \text{V} \\
& \leq -\frac{1}{2} e^{-\beta t} \int_M \phi^2 e^h f_K^2 d\mu_t|_{t=\eta} + \frac{1}{2} e^{-\beta t} \int_M \phi^2 e^h f_K^2 d\mu_t|_{t=\epsilon} \\
& + \frac{1}{2} \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h f_K^2 \frac{\partial h}{\partial t} d\mu_t \right) dt - \frac{1}{4} \beta \int_{\epsilon}^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h f_K^2 d\mu_t \right) dt \\
& + e^{-\beta t} \int_M \phi^2 e^h f_K (f_K - f) d\mu_t|_{t=\eta} - e^{-\beta t} \int_M \phi^2 e^h f_K^2 d\mu_t|_{t=\epsilon}.
\end{aligned}$$

Combining the estimates of I – VI and letting $\epsilon \rightarrow 0$, we get

$$\begin{aligned}
& - \int_0^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h a |\nabla f_K|^2 d\mu_t \right) dt + \int_0^{\eta} e^{-\beta t} \left(\int_M \phi^2 e^h a |\nabla f|^2 d\mu_t \right) dt \\
& + 2 \int_0^{\eta} e^{-\beta t} \left(\int_M e^h f_K^2 a |\nabla \phi|^2 d\mu_t \right) dt - \frac{1}{2} e^{-\beta t} \int_M \phi^2 e^h f_K^2 d\mu_t|_{t=\eta} \geq 0.
\end{aligned}$$

by $f_K \equiv 0$ at $t = 0$ and (6.1). Now we choose $0 \leq \phi \leq 1$ satisfying $\phi \equiv 1$ on $B_{g_0}(p, R)$, $\phi \equiv 0$ outside $B_{g_0}(p, R+1)$ and $|\nabla_{g_0} \phi|_{g_0} \leq 2$. Then we have

$$\begin{aligned}
& \frac{1}{2} e^{-\beta \eta} \int_{B_{g_0}(p, R)} \phi^2 e^h f_K^2 d\mu_t|_{t=\eta} \leq \int_0^{\eta} e^{-\beta t} \left(\int_{B_{g_0}(p, R+1)} \phi^2 e^h a (|\nabla f|^2 - |\nabla f_K|^2) d\mu_t \right) dt \\
& + C(\alpha_5) \int_0^{\eta} e^{-\beta t} \left(\int_{B_{g_0}(p, R+1) \setminus B_{g_0}(p, R)} e^h f_K^2 a d\mu_t \right) dt,
\end{aligned}$$

where $C(\alpha_5)$ is a constant only depending on α_5 . By $0 < \eta < \min(\frac{1}{K}, \frac{1}{32\alpha_4})$ and volume growth assumptions on M^n , we have

$$\int_0^{\eta} e^{-\beta t} \left(\int_{B_{g_0}(p, R+1) \setminus B_{g_0}(p, R)} e^h f_K^2 a d\mu_t \right) dt \rightarrow 0,$$

as $R \rightarrow \infty$. Then we derive

$$\frac{1}{2}e^{-\beta\eta} \int_M \phi^2 e^h f_K^2 d\mu_t|_{t=\eta} \leq \int_0^\eta e^{-\beta t} \left(\int_M \phi^2 e^h a(|\nabla f|^2 - |\nabla f_K|^2) d\mu_t \right) dt.$$

Now letting $K \rightarrow \infty$, we conclude that

$$\frac{1}{2}e^{-\beta\eta} \int_M \phi^2 e^h (\max(f, 0))^2 d\mu_t|_{t=\eta} \leq 0,$$

where $0 < \eta < \min(T, \frac{1}{64K}, \frac{1}{32\alpha_4}, \frac{1}{4\alpha_5})$. By the inductive argument, we conclude that $f \leq 0$ in $M^n \times [0, T]$. \square

Acknowledgement: We would like thank Prof. Li Ma bring this problem to us. The first author would like thank Doctor Y.F.Chen for useful talking about the paper [5].

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